ME 7247: Advanced Control Systems **Supplementary notes**

The bounded real lemma

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1 Boundedness

There are several different equivalent ways of characterizing the boundedness of a linear dynamical system,in the sense of "bounded input, bounded output". We consider the dynamical system:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t + Du_t \end{aligned} \tag{1}$$

Theorem 1. *Suppose (A, B, C, D) is a minimal realization, so (A, B) is controllable and (A, C) is observable. The following statements are equivalent.*

(i) Let $\{u_0, u_1, \dots\}$ and $\{y_0, y_1, \dots\}$ be any sequence of inputs and outputs that satisfy (1) with $x_0 = 0$. The system has gain bound γ, which means that whenever $u \in \ell_2$, we have

$$\|y\| \le \gamma \|u\|.$$

(ii) Let $\{u_0, u_1, \dots,\}$ and $\{y_0, y_1, \dots\}$ be any sequence of inputs and outputs that satisfy (1) with $x_0 = 0$. The system has finite gain bound γ, which means that

$$\sum_{t=0}^{N-1} \|y_t\|^2 \le \gamma^2 \sum_{t=0}^{N-1} \|u_t\|^2 \qquad \textit{for all } N.$$

(iii) $F_N(\xi) \ge 0$ for all ξ and all N, where F_N is defined as

$$\begin{aligned} F_N(\xi) \quad := \quad & \underset{\substack{u_0,\dots,u_{N-1} \\ y_0,\dots,y_{N-1} \\ x_0,\dots,x_N}}{\text{minimize}} \quad \sum_{t=0}^{N-1} \left(\gamma^2 \|u_t\|^2 - \|y_t\|^2\right) \\ & \text{s.t.} \quad x_{t+1} = Ax_t + Bu_t, \\ & \qquad\;\; y_t = Cx_t + Du_t \quad \text{for } t = 0, \dots, N-1 \\ & \qquad\;\; x_0 = 0, \quad x_N = \xi \end{aligned}$$

(iv) There exists a matrix $P \succ 0$ satisfying the following LMI.

$$\begin{bmatrix} A^\mathsf{T}PA - P + C^\mathsf{T}C & A^\mathsf{T}PB + C^\mathsf{T}D \\ B^\mathsf{T}PA + D^\mathsf{T}C & B^\mathsf{T}PB + D^\mathsf{T}D - \gamma^2 I \end{bmatrix} \preceq 0$$

(v) There exists a function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$ such that for all $\{x_t, u_t, y_t\}$ that satisfy (1), we have the following dissipation inequality*.*

$$V(x_{t+1}) - V(x_t) \le \gamma^2 \|u_t\|^2 - \|y_t\|^2.$$

Proof. We will prove Theorem 1 by proving (i) $\iff$ (ii) $\implies$ (iii) $\implies$ (iv) $\implies$ (v) $\implies$ (ii).

(i) $\implies$ (ii). Suppose (i) holds. Let $x_0 = 0$ and let $\{u_0, u_1, \dots\}$ and $\{y_0, y_1, \dots\}$ be inputs and outputs that satisfy (1). Define $\hat{u}$ and $\hat{y}$ to be the truncated versions of these signals:

$$\hat{u}_t := \begin{cases} u_t & 0 \le t \le N-1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \hat{y}_t := \begin{cases} y_t & 0 \le t \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

Since the system (1) is causal, applying the input $\hat{u}$ actually produces $\hat{y}$ as an output. Now write

$$\sum_{t=0}^{N-1} \|y_t\|^2 = \sum_{t=0}^{\infty} \|\hat{y}_t\|^2 \le \gamma^2 \sum_{t=0}^{\infty} \|\hat{u}_t\|^2 = \gamma^2 \sum_{t=0}^{N-1} \|u_t\|^2.$$

The inequality in the middle follows from applying Item (i) to the signals $\hat{u}$ and $\hat{y}$. Note that $\hat{u} \in \ell_2$ since it consists of finitely many nonzero components. ■

(ii) $\implies$ (i). Suppose (ii) holds. Let $x_0 = 0$ and let $\{u_0, u_1, \dots\}$ and $\{y_0, y_1, \dots\}$ be inputs and outputs that satisfy (1). If $u \in \ell_2$, then we have

$$\sum_{t=0}^{N-1} \|y_t\|^2 \le \gamma^2 \sum_{t=0}^{N-1} \|u_t\|^2 \le \gamma^2 \sum_{t=0}^{\infty} \|u_t\|^2 = \gamma^2 \|u\|^2.$$

The left-hand side is an increasing function of N and uniformly bounded above, so the limit $N \to \infty$ exists, and we conclude that $y \in \ell_2$ and $\|y\|^2 \le \gamma^2 \|u\|^2$, as required. ■

(ii) $\implies$ (iii). Nonnegativity of the objective function follows immediately from (ii), so the optimization problem must be nonnegative for every ξ. Note that if the optimization problem is infeasible, we have $F(\xi) = \infty \ge 0$ so nonnegativity still holds. ■

(iii) $\implies$ (iv). Suppose that Item (iii) holds. The function $F_N(\xi)$ has many useful properties. First, F_N is quadratic whenever $N \ge n$. This follows from the fact that optimizing a quadratic function subject to linear constraints is quadratic whenever it is finite. To check finiteness, first we have $F_N(\xi) \ge 0$ so the problem is bounded below. Next, the problem is feasible for $N \ge n$ due to controllability of (A, B), so $F_N(\xi) < \infty$. The problem is therefore finite, and we can write $F_N(\xi) = \xi^\mathsf{T} P_N \xi$ for some matrix $P_N \succeq 0$.

Next, $F_N(\xi)$ is monotonically nonincreasing in N. This is because if a particular optimal cost can be attained for some N, it can also be attained for any $\hat{N} > N$ by picking $u_N = \cdots = u_{\hat{N}-1} = 0$, as the state will remain at $x_N = \cdots = x_{\hat{N}} = 0$. We conclude that $P_{\hat{N}} \preceq P_N$ whenever $\hat{N} \ge N$.

Since $F_N(\xi)$ is bounded below and monotonically nonincreasing, it must tend to a limit. Therefore, we have $\lim_{N\to\infty} F_N(\xi) = F(\xi)$. Since F_N is quadratic for each N, the limit is also quadratic, and we conclude that $\lim_{N\to\infty} P_N = P$ and $F(\xi) = \xi^\mathsf{T} P \xi$ with $P \succeq 0$.

We will now bound F_N in terms of F_{N-1} using a dynamic programming-like argument. Let ξ be any state and η be any input.

$$\begin{aligned}
F_N(A\xi + B\eta) \quad &= \quad \underset{\substack{u_0,\dots,u_{N-1}\\ y_0,\dots,y_{N-1}\\ x_0,\dots,x_N}}{\text{minimize}} \quad \sum_{k=0}^{N-1} \left(\gamma^2\|u_t\|^2 - \|y_t\|^2\right) \\
&\qquad\qquad \text{s.t.} \quad x_{t+1} = Ax_t + Bu_t, \\
&\qquad\qquad\qquad\; y_t = Cx_t + Du_t \quad \text{for } t = 0,\dots,N-1 \\
&\qquad\qquad\qquad\; x_0 = 0, \quad x_N = A\xi + B\eta \\
&\leq \quad \underset{\substack{u_0,\dots,u_{N-1}\\ y_0,\dots,y_{N-1}\\ x_0,\dots,x_N}}{\text{minimize}} \quad \sum_{k=0}^{N-1} \left(\gamma^2\|u_t\|^2 - \|y_t\|^2\right) \\
&\qquad\qquad \text{s.t.} \quad x_{t+1} = Ax_t + Bu_t, \\
&\qquad\qquad\qquad\; y_t = Cx_t + Du_t \quad \text{for } t = 0,\dots,N-1 \\
&\qquad\qquad\qquad\; x_0 = 0, \quad x_{N-1} = \xi, \quad u_{N-1} = \eta \\
&= \quad \underset{\substack{u_0,\dots,u_{N-2}\\ y_0,\dots,y_{N-2}\\ x_0,\dots,x_{N-1}}}{\text{minimize}} \quad \sum_{k=0}^{N-2} \left(\gamma^2\|u_t\|^2 - \|y_t\|^2\right) + \left(\gamma^2\|\eta\|^2 - \|C\xi + D\eta\|^2\right) \\
&\qquad\qquad \text{s.t.} \quad x_{t+1} = Ax_t + Bu_t, \\
&\qquad\qquad\qquad\; y_t = Cx_t + Du_t \quad \text{for } t = 0,\dots,N-2 \\
&\qquad\qquad\qquad\; x_0 = 0, \quad x_{N-1} = \xi \\
&= \quad F_{N-1}(\xi) + \gamma^2\|\eta\|^2 - \|C\xi + D\eta\|^2
\end{aligned}$$

Taking the limit $N \to \infty$, we obtain the inequality:

$$F(A\xi + B\eta) \leq F(\xi) + \gamma^2\|\eta\|^2 - \|C\xi + D\eta\|^2$$

We previously established that $F(x) = x^\mathsf{T} P x$ with $P \succeq 0$. Substituting into the above, we obtain

$$(A\xi + B\eta)^\mathsf{T} P(A\xi + B\eta) - \xi^\mathsf{T} P\xi + (C\xi + D\eta)^\mathsf{T}(C\xi + D\eta) - \gamma^2\eta^\mathsf{T}\eta \leq 0$$

Write the left-hand side as a quadratic form and obtain:

$$\begin{bmatrix}\xi\\ \eta\end{bmatrix}^\mathsf{T} \begin{bmatrix} A^\mathsf{T}PA - P + C^\mathsf{T}C & A^\mathsf{T}PB + C^\mathsf{T}D \\ B^\mathsf{T}PA + D^\mathsf{T}C & B^\mathsf{T}PB + D^\mathsf{T}D - \gamma^2 I \end{bmatrix} \begin{bmatrix}\xi\\ \eta\end{bmatrix} \leq 0$$

this must hold for all (ξ, η), so we obtain Item (iv), as required. To prove that $P \succ 0$, the $(1,1)$ block implies that $A^\mathsf{T}PA - P + C^\mathsf{T}C \preceq 0$. This means there must exist some matrix $W \succ 0$ such that $A^\mathsf{T}PA - P + C^\mathsf{T}C + W = 0$. Since $W \succeq 0$, we can factor $W = H^\mathsf{T}H$ and rewrite as:

$$A^\mathsf{T}PA - P + \begin{bmatrix}C\\ H\end{bmatrix}^\mathsf{T} \begin{bmatrix}C\\ H\end{bmatrix} = 0$$

This is a Lyapunov equation with $P \succeq 0$ and (A, C) observable. Therefore, $(A, \left[\begin{smallmatrix}C\\ H\end{smallmatrix}\right])$ is observable, and we conclude that A is Schur-stable and $P \succ 0$. ■

(iv) $\Longrightarrow$ (v). Suppose (iv) holds. Multiply both sides by (x_t, u_t) and substitute the dynamics (1):

$$x_{t+1}^\mathsf{T} P x_{t+1} - x_t^\mathsf{T} P x_t \leq \gamma^2\|u_t\|^2 - \|y_t\|^2.$$

Letting $V(x) := x^\mathsf{T} P x$, the inequality above becomes Item (v). The fact that $P \succ 0$ implies that $V(x) > 0$ for all $x \neq 0$ and $V(0) = 0$, as required. ■

(v) $\implies$ (ii). Suppose (v) holds and $x_0 = 0$. Sum the dissipation inequality from $t = 0$ to $t = N-1$ and use the fact that $V(x_0) = V(0) = 0$ to obtain

$$V(x_N) \le \sum_{t=0}^{N-1} \left(\gamma^2 \|u_t\|^2 - \|y_t\|^2\right).$$

Since V is positive definite, the left-hand side is nonnegative. Rearranging, we obtain (ii). ■

Remark 1. In the proof of Theorem 1, the controllability assumption is only used in (iii) $\implies$ (iv) and the observability assumption is only used in proving that $P \succ 0$ in refbropt $\implies$ (iv). If we remove the observability assumption, we still have $P \succeq 0$.

There are many equivalent ways of writing the LMI from Item (iv) of Theorem 1. These follow from applying properties of the Schur complement and positive definite matrices.

Corollary 1 (Alternative LMIs)**.** *The following statements are equivalent.*

(i) There exists $P \succ 0$ such that

$$\begin{bmatrix} A^\mathsf{T} PA - P + C^\mathsf{T} C & A^\mathsf{T} PB + C^\mathsf{T} D \\ B^\mathsf{T} PA + D^\mathsf{T} C & B^\mathsf{T} PB + D^\mathsf{T} D - \gamma^2 I \end{bmatrix} \preceq 0.$$

(ii) There exists $P \succ 0$ such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^\mathsf{T} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & \gamma^2 I \end{bmatrix} \preceq 0.$$

(iii) There exists $P \succ 0$ such that

$$\begin{bmatrix} A^\mathsf{T} PA - P & A^\mathsf{T} PB & C^\mathsf{T} \\ B^\mathsf{T} PA & B^\mathsf{T} PB - \gamma I & D^\mathsf{T} \\ C & D & -\gamma I \end{bmatrix} \preceq 0.$$

(iv) There exists $P \succ 0$ such that

$$\begin{bmatrix} P & PA & PB & 0 \\ A^\mathsf{T} P & P & 0 & C^\mathsf{T} \\ B^\mathsf{T} P & 0 & \gamma I & D^\mathsf{T} \\ 0 & C & D & \gamma I \end{bmatrix} \succeq 0.$$

Remark 2. We can also set $Q = P^{-1}$ and rearrange the LMIs in Corollary 1 to be linear in Q instead. This yields a dual set of analogous LMIs. Practically speaking, this is exactly equivalent to taking any of the LMIs in Corollary 1 and performing the change of variables

$$(P, A, B, C, D) \mapsto (Q, A^\mathsf{T}, C^\mathsf{T}, B^\mathsf{T}, D^\mathsf{T}).$$

This is a manifestation of the fact that a system G and its transpose G^T have the same $\mathcal{H}_\infty$-norm. It is also analogous to the dual representations we found for the $\mathcal{H}_2$ norm, which demonstrate the similar fact that G and G^T also have the same $\mathcal{H}_2$-norm.

2 The bounded real lemma

The name *bounded real lemma* typically refers to an equivalence between the LMI of Theorem 1 and a frequency-domain condition. Here is the result.

Theorem 2 (Bounded real lemma). *Let $G(z) := C(zI - A)^{-1}B + D$, where (A, B, C, D) is a minimal realization. The following statements are equivalent.*

(i) There exists a matrix $P \succ 0$ satisfying the following LMI.

$$\begin{bmatrix} A^\mathsf{T}PA - P + C^\mathsf{T}C & A^\mathsf{T}PB + C^\mathsf{T}D \\ B^\mathsf{T}PA + D^\mathsf{T}C & B^\mathsf{T}PB + D^\mathsf{T}D - \gamma^2 I \end{bmatrix} \preceq 0 \tag{2}$$

(ii) For all $z \in \mathbb{C}$ such that $|z| \geq 1$, the following frequency-domain inequality holds.

$$G(z)^* G(z) \preceq \gamma^2 I. \tag{3}$$

Proof. **Proof that (i) $\implies$ (ii).** Suppose (i) holds. Pick z such that $\det(zI - A) \neq 0$, so $zI - A$ is invertible. Start with (2) and compute

$$\begin{aligned} &\begin{bmatrix} (zI-A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} A^\mathsf{T}PA - P + C^\mathsf{T}C & A^\mathsf{T}PB + C^\mathsf{T}D \\ B^\mathsf{T}PA + D^\mathsf{T}C & B^\mathsf{T}PB + D^\mathsf{T}D - \gamma^2 I \end{bmatrix} \begin{bmatrix} (zI-A)^{-1}B \\ I \end{bmatrix} \preceq 0 \\ \iff \quad &\begin{bmatrix} (zI-A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} A^\mathsf{T}PA - P & A^\mathsf{T}PB \\ B^\mathsf{T}PA & B^\mathsf{T}PB \end{bmatrix} \begin{bmatrix} (zI-A)^{-1}B \\ I \end{bmatrix} + G(z)^*G(z) \preceq \gamma^2 I \end{aligned}$$

The term on the left simplifies to

$$\begin{aligned} &\begin{bmatrix} (zI-A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} A^\mathsf{T}PA - P & A^\mathsf{T}PB \\ B^\mathsf{T}PA & B^\mathsf{T}PB \end{bmatrix} \begin{bmatrix} (zI-A)^{-1}B \\ I \end{bmatrix} \\ &\quad = \begin{bmatrix} (zI-A)^{-1}B \\ I \end{bmatrix}^* \left(\begin{bmatrix} A^\mathsf{T} \\ B^\mathsf{T} \end{bmatrix} P \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} (zI-A)^{-1}B \\ I \end{bmatrix} \\ &\quad = \Big(B^\mathsf{T}(\bar{z}I - A^\mathsf{T})^{-1}A^\mathsf{T} + B^\mathsf{T} \Big) P \Big(A(zI-A)^{-1}B + B \Big) - B^\mathsf{T}(\bar{z}I - A^\mathsf{T})^{-1}P(zI-A)^{-1}B \\ &\quad = B^\mathsf{T}(\bar{z}I - A^\mathsf{T})^{-1}\left(\bar{z}zP - P\right)(zI-A)^{-1}B = 0 \\ &\quad = (|z|^2 - 1) \cdot B^\mathsf{T}(\bar{z}I - A^\mathsf{T})^{-1}P(zI-A)^{-1}B = 0 \\ &\quad \succeq 0. \end{aligned}$$

In the last step, we used the fact that $|z|^2 \geq 1$ and $P \succ 0$. Therefore (3) holds and hence we have proven Item (ii), as required.

Proof that (ii) $\implies$ (i). Suppose (ii) holds. Let $u \in \ell_2$ and consider its z-transform $\hat{u}(z)$. Then the output of the system has z-transform $\hat{y}(z) = G(z)\hat{u}(z)$. Starting with 3, we have

$$\hat{y}(z)^*\hat{y}(z) = \hat{u}(z)^*G(z)^*G(z)\hat{u}(z) \preceq \gamma^2 \hat{u}(z)^*\hat{u}(z)$$

Integrating both sides along the unit circle, we obtain:

$$\int_{-\pi}^{\pi} \hat{y}(e^{i\theta})^*\hat{y}(e^{i\theta})\,\mathrm{d}\theta \leq \gamma^2 \int_{-\pi}^{\pi} \hat{u}(e^{i\theta})^*\hat{u}(e^{i\theta})\,\mathrm{d}\theta$$

The integral on the right-hand side converges, because $u \in \ell_2$, which implies $\hat{u} \in \ell_2$. The integral on the left-hand side is bounded above and its integrand is nonnegative, so the integral must also converge, and we have $\hat{y} \in \ell_2$. Apply the discrete version of Parseval's theorem and obtain

$$\int_0^\infty y(t)^\mathsf{T} y(t)\,\mathrm{d}t \le \gamma^2 \int_0^\infty u(t)^\mathsf{T} u(t)\,\mathrm{d}t.$$

In other words, $\|y\| \le \gamma\|u\|$ for all $u \in \ell_2$, so G has gain bound γ. We can now apply Theorem 1 to prove that the LMI (2) holds. ■

Remark 3. There are points at which $G(z)$ is undefined, namely whenever $zI - A$ is not invertible. These are the poles of $G(z)$. We don't need to worry about such points in Item (ii) of Theorem 2 because if $G(z)$ had a pole satisfying $|z| \ge 1$, then $\operatorname{trace}(G(z)^* G(z))$ would approach $+\infty$ near that pole, and so (3) could not hold for any finite γ. In other words, if Item (ii) holds, then G must be a stable transfer matrix.

Remark 4. If we replace the $\preceq$ symbols in (2) and (3) with $\prec$, it is possible to prove Theorem 2 without the need for the minimality assumption on (A, B, C, D). The proof method is different, however, since we can no longer use Theorem 1.

Theorem 2 provides the following frequency-domain characterization of the $\mathcal{H}_\infty$-norm.

Corollary 2. *Suppose G is a linear system with transfer function $G(z)$. We have the following equivalent characterizations of the $\mathcal{H}_\infty$ norm.*

$$\|G\|_\infty = \sup_{\substack{u \in \ell_2 \\ u \neq 0}} \frac{\|Gu\|}{\|u\|} = \sup_{|z|>1} \|G(z)\|$$

If we further assume that G is stable to begin with, so it has no poles in the closed right-half plane, we can apply the maximum modulus principle and deduce that:

$$\|G\|_\infty = \sup_{|z|=1} \|G(z)\| = \sup_{\theta \in [-\pi,\pi]} \|G(e^{i\theta})\|$$

This is more practical because it is often easy to check stability, and then we can turn the optimization over the region $|z| > 1$ into an optimization over the compact interval $\theta \in [-\pi, \pi]$. Using this interpretation, we see that when G is a stable SISO system (single-input, single-output), $\|G\|_\infty$ is the peak of the Bode magnitude plot of G.