ME 7247: Advanced Control Systems **Supplementary notes**

Lyapunov functions

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1 Problem setting and statement

Let $x_t \in \mathbb{R}^n$. Consider a dynamical system of the following form and define some new terms.

$$x_{t+1} = f(x_t) \tag{1}$$

- ***Fixed point:*** We say that $x_\star$ is a *fixed point* of Eq. (1) if $f(x_\star) = x_\star$. So if x_t ever reaches $x_\star$, it will stay there for all subsequent t.
- ***Global asymptotic stability:*** We say that the point $x_\star$ is globally asymptotically stable if for any $x_0 \in \mathbb{R}^n$, we have $\lim_{t\to\infty} x_t = x_\star$.
- ***Class KR function:*** We say that a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is of class KR, written $\phi \in KR$, if ϕ is continuous, strictly increasing, and $\phi(0) = 0$. Additionally, ϕ is *radially unbounded*, i.e., $\lim_{p\to\infty} \phi(p) = \infty$. Class KR functions are invertible $\phi \in KR \iff \phi^{-1} \in KR$.

Here is a global asymptotic stability result.

Theorem 1. *Consider the dynamical system* (1)*. Suppose f is continuous, the origin is a fixed point, and there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfying*

(i) $V(0) = 0$ *and there exists* $\phi \in KR$ *such that for all* $x \in \mathbb{R}^n$*, we have* $V(x) \geq \phi(\|x\|)$.

(ii) $V(f(x)) - V(x) < 0$ *for all* $x \neq 0$.

Then, the origin is globally asymptotically stable

A function V satisfying the conditions in Theorem 1 is called a ***Lyapunov function***.

This approach of finding a Lyapunov function V to prove stability of a dynamical system is called *Lyapunov's direct method* (a.k.a. Lyapunov's second method). This is in contrast with *Lyapunov's indirect method* (a.k.a. Lyapunov's first method), which is a way of deducing stability of a nonlinear system by studying properties of its linearization.

Before we prove Theorem 1, we will need two important results from real analysis.

Lemma 1 (Monotone Convergence Theorem)**.** *Consider the sequence* $\{x_0, x_1, \dots\} \subseteq \mathbb{R}$.

- *If* $x_0 \geq x_1 \geq \dots$ *(monotonically decreasing), and* $x_k \geq a$ *for all* k *(bounded below), then the sequence converges, and* $\lim_{t\to\infty} x_t = \inf_{t\geq 0} x_t$.
- *If* $x_0 \leq x_1 \leq \dots$ *(monotonically increasing), and* $x_k \leq b$ *for all* k *(bounded above), then the sequence converges, and* $\lim_{t\to\infty} x_t = \sup_{t\geq 0} x_t$.

The Monotone Convergence Theorem (MCT) is a sufficient condition for the existence of a limit. Of course, it is possible for non-monotone sequences to have limits.

Lemma 2 (Extreme Value Theorem)**.** *Suppose $S \in \mathbb{R}^n$ is closed and bounded and $f : S \to \mathbb{R}$ is continuous. Then f achieves a minimum and a maximum on S.*

To understand how the Extreme Value Theorem (EVT) manifests itself, consider $f(x) = 1/x$.

- If we pick $S = (0, 1)$, then f has no maximum on S; we can make $f(x)$ as large as we like by picking x closer to 0. S is not closed so the EVT does not apply.
- If we pick $S = [1, \infty)$, then f has no minimum on S; we can $f(x)$ as small as we like by picking larger x. S is not bounded so the EVT does not apply.
- If we pick $S = [-1, 1]$, then f has no minimum or maximum on S because of the asymptote at $x = 0$. f is not continuous on S, so the EVT does not apply.
- If we pick $S = [1, 2]$, then f has both a minimum and a maximum. The EVT applies because S is closed and bounded, and f is continuous on S.

The EVT is only a sufficient condition for the existence of extrema. For example, it's possible to violate all the requirements of EVT and still attain a maximum and/or minimum value.

2 Proof of Theorem 1

From Item (ii), the sequence $V(x_0), V(x_1), V(x_2), \dots$ is a decreasing sequence and from Item (i), it is bounded below by zero. By the MCT, this sequence converges to a limit. Let $\alpha := \lim_{t\to\infty} V(x_t)$. We already know $\alpha \ge 0$. We will now prove that $\alpha = 0$ by way of contradiction.

Suppose that $\alpha > 0$. Consider the set $S := \{z \in \mathbb{R}^n \mid \alpha \le V(z) \le V(x_0)\}$. This set is closed and bounded. To see why, we have from Item (i) that $V(z) \le V(x_0)$ implies $\phi(\|z\|) \le V(x_0)$ and therefore $\|z\| \le \phi^{-1}(V(x_0))$. So S is contained inside a ball of radius $\phi^{-1}(V(x_0))$ and is therefore bounded.[1] S is closed because both endpoints in S are included and V is a continuous function. Apply the EVT to the continuous function $V(f(z)) - V(z)$ over the set S.[2] Let the maximum of this function over S be β. Since $\alpha > 0$, all points in S satisfy $V(z) > 0$, and therefore $z \neq 0$. Since Item (ii) holds strictly, we conclude that $\beta < 0$. For any $t \ge 0$, we have:

$$V(x_t) = V(x_0) + \sum_{k=1}^{t} \big(V(x_k) - V(x_{k-1})\big) \le V(x_0) + t\beta \tag{2}$$

Since $\beta < 0$, if we make t sufficiently large, we can make the right-hand side of Eq. (2) negative, which would imply that $V(x_t) < 0$, a contradiction. Consequently, our supposition that $\alpha > 0$ was incorrect, and therefore $\alpha = 0$, and $\lim_{t\to\infty} V(x_t) = 0$. By the definition of the limit, we have that for any $\varepsilon > 0$, there exists a $T > 0$ such that for all $t \ge T$, we have $V(x_t) < \phi(\varepsilon)$. Applying Item (i), this means $\phi(\|x_t\|) < \phi(\varepsilon)$, and therefore $\|x_t\| < \varepsilon$. Consequently $\lim_{t\to\infty} x_t = 0$. ∎

[1] This is where we use the fact that ϕ is radially unbounded. Without this, ϕ may not be invertible, and we could end up with an unbounded S.

[2] Here, we used the fact that f and V are continuous, therefore $V(f(z)) - f(z)$ is a continuous function of z.

3 Exponential stability

We say a fixed point $x_\star$ of Eq. (1) is ***globally exponentially stable*** if there exists $c > 0$ and $\rho \in (0,1)$ such that for any initial condition $x_0 \in \mathbb{R}^n$,

$$\|x_t - x_\star\| \leq c\,\|x_0 - x_\star\|\,\rho^t \qquad \text{for all } t \geq 0.$$

Global exponential stability is stronger than global asymptotic stability, because not only do we have $\lim_{t\to\infty} x_t = x_\star$, but we also guarantee convergence at an exponential rate. A sequence such as $x_t = 1/t$ converges to zero asymptotically, but not exponentially.

We can modify the assumptions of Theorem 1 to achieve exponential convergence.

Theorem 2. *Consider the dynamical system* (1). *Suppose the origin is a fixed point, and there exists a function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *satisfying*

(i) There exists $p > 0$ *and* $\alpha, \beta > 0$ *such that* $\alpha\|x\|^p \leq V(x) \leq \beta\|x\|^p$ *for all* $x \in \mathbb{R}^n$.

(ii) There exists $0 < \rho < 1$ *such that* $V(f(x)) - \rho V(x) \leq 0$ *for all* $x \neq 0$.

Then, the origin is globally exponentially stable

Proof. Applying Items (i) and (ii), we have:

$$\alpha\|x_t\|^p \leq V(x_t) \leq \rho V(x_{t-1}) \leq \cdots \leq \rho^t V(x_0) \leq \beta\|x_0\|^p \rho^t.$$

Rearranging this inequality, we obtain

$$\|x_t\| \leq \left(\frac{\beta}{\alpha}\right)^{1/p} \|x_0\| \left(\rho^{1/p}\right)^t,$$

so the origin is globally exponentially stable. ∎