ME 7247: Advanced Control Systems **Fall 2022–23**

Lecture 22: $\mathcal{H}_2$ and $\mathcal{H}_\infty$ infinity control

Tuesday November 29, 2022

Lecturer: Laurent Lessard *Scribe: Mayur Bhise*

In this lecture, we study two pillars of controls: $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control. We frame LQR and LQG as types of $\mathcal{H}_2$ control. We also cover the bounded real lemma, an important ingredient in $\mathcal{H}_\infty$ control.

1 System norms

We previously discussed vector norms and matrix norms. We will review these concepts and add one more; the concept of a system norm.

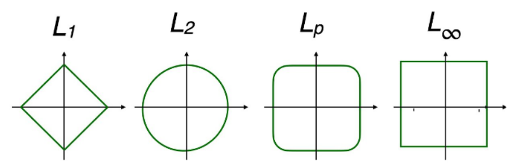
1.1 Vector Norms

L_1 norm: $$\|x\|_1 = \sum_i |x_i|$$

L_2 norm: $$\|x\| = \|x\|_2 = \sqrt{\sum_i x_i^2}$$

L_p norm: $$\|x\|_1 = \left(\sum_i |x_i|^p\right)^{1/p}$$

L_∞ norm: $$\|x\|_\infty = \max_i |x_i|$$



1.2 Matrix Norms

- Spectral norm / induced 2-norm / matrix norm: $\|A\| = \|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \sigma_1(A)$
- Frobenius norm : $\|A\|_F = \sum_{ij} |A_{ij}|^2 = \mathbf{tr}(A^\mathsf{T} A) = \sum_i \sigma_i(A)^2$

1.3 System Norms

$$u \longrightarrow \boxed{\begin{aligned} x_{t+1} &= A_t x_t + B u_t \\ y_t &= C x_t \end{aligned}} \longrightarrow y \tag{1}$$

- The $\mathcal{H}_2$ norm squared is $\|G\|_2^2 = \sum_{t=0}^{\infty} \|Y_t\|_F^2$, where either $U_0 = I$ and $U_t = 0$ for $t \geq 1$ (Frobenius norm of the matrix impulse response), or it is the steady-state average covariance when the input is white noise. In other words, $\|G\|_2^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} \mathbb{E}\|y_t\|^2$, where $u_t \sim \mathcal{N}(0, I)$. Both definitions are equivalent.

- The $\mathcal{H}_\infty$ norm is the maximum gain of the system G when the input is square-summable. In other words, $\|G\|_\infty = \sup_{u\in\ell_2} \frac{\|Gu\|}{\|u\|}$.

We can view $\|G\|_2$ and $\|G\|_\infty$ as analogous to $\|A\|_F$ and $\|A\|$ for matrices, respectively. It is unfortunate that the naming conventions are not more consistent. The spectral norm for matrices is often written as $\|A\|_2$ whereas the corresponding norm for systems is $\|G\|_\infty$. Meanwhile, $\|G\|_2$ for systems means something completely different, and is not an induced norm at all.

2 The $\mathcal{H}_2$ norm

2.1 Impulse response

The (matrix) impulse response of the system assumes matrix inputs of $U_0 = I$ and $U_t = 0$ for $t \ge 1$, with an initial state of $X_0 = 0$. This yields matrix outputs of:

$$Y_0 = 0, \quad Y_1 = CB, \quad Y_2 = CAB, \quad Y_3 = CA^2B, \dots \quad Y_k = CA^{k-1}B$$

Therefore, the $\mathcal{H}_2$ norm of a system is given by:

$$\begin{aligned}\|G\|_2^2 &= \mathbf{tr}\left(\begin{bmatrix} CB \\ CAB \\ CA^2B \\ \vdots \end{bmatrix}^\mathsf{T} \begin{bmatrix} CB \\ CAB \\ CA^2B \\ \vdots \end{bmatrix}\right) = \mathbf{tr}\left(B^\mathsf{T}C^\mathsf{T}CB + B^\mathsf{T}A^\mathsf{T}C^\mathsf{T}CAB + B^\mathsf{T}(A^\mathsf{T})^2C^\mathsf{T}CA^2B + \dots\right) \\ &= \mathbf{tr}\left(B^\mathsf{T}(C^\mathsf{T}C + A^\mathsf{T}C^\mathsf{T}CA + (A^\mathsf{T})^2C^\mathsf{T}CA^2 + \dots)B\right)\end{aligned}$$

Call the term inside the inner bracket Q, and note that Q satisfies a Lyapunov equation. In fact, Q is the observability Gramian of G. We conclude that

$$\|G\|_2^2 = \mathbf{tr}(B^\mathsf{T}QB), \qquad \text{where } Q \text{ satisfies} \quad A^\mathsf{T}QA - Q + C^\mathsf{T}C = 0$$

Using the fact that $\mathbf{tr}(XY) = \mathbf{tr}(YX)$, write $\mathbf{tr}(B^\mathsf{T}(A^\mathsf{T})^kC^\mathsf{T}CA^kB) = \mathbf{tr}(CA^kBB^\mathsf{T}(A^\mathsf{T})^kC^\mathsf{T})$. This leads us to an alternative but equivalent definition in terms of the controllability Gramian:

$$\|G\|_2^2 = \mathbf{tr}(CPC^\mathsf{T}), \qquad \text{where } P \text{ satisfies} \quad APA^\mathsf{T} - P + BB^\mathsf{T} = 0$$

2.2 LQR interpretation

Recall the LQR problem is to find inputs $\{u_0, u_1, \dots\}$ that minimize the cost

$$J = \sum_{t=0}^{\infty} x_t^\mathsf{T}Qx_t + u_t^\mathsf{T}Ru_t$$

where $x_{t+1} = Ax_t + Bu_t$. We can view this as a case of minimizing $\|y\|^2$, where we defined the output signal as

$$y_t = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u_t = Fx_t + Hu_t$$

When using the control law $u_t = Kx_t$, we can rewrite the closed-loop system G_{CL} as:

$$\begin{aligned} x_{t+1} &= (A + BK)x_t + w_t \\ y_t &= \begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix} x_t \end{aligned}$$

Now consider the $\mathcal{H}_2$ norm of this system. specifically, imagine that $w_{-1} = x_0$ and we let $w_t = 0$ for all future timesteps. Then using the observability Gramian formulation (renaming Q to X), we have

$$\|G_{\text{CL}}\|_2^2 = x_0^\mathsf{T} X x_0, \qquad \text{where } X \text{ satisfies} \quad (A - BK)^\mathsf{T} X (A - BK) - X + (Q + K^\mathsf{T} RK) = 0$$

Note that when K is the optimal LQR gain, the above Lyapunov equation becomes the DARE for the LQR problem. Therefore, we can interpret LQR control as finding the controller that minimizes the $\mathcal{H}_2$ norm of the closed-loop map from w to y.

2.3 LQG interpretation

We omit the details, but the LQG interpretation is similar to the LQR interpretation. The optimal LQG controller also minimizes the $\mathcal{H}_2$ norm of the closed-loop map from the external disturbance w to the performance output channel.

3 The $\mathcal{H}_\infty$ norm

The $\mathcal{H}_\infty$ is a *worst-case* norm. It is defined as

$$\|G\|_\infty = \sup_{u \in \ell_2} \frac{\|Gu\|}{\|u\|}$$

We can view this as the matrix norm of the infinite-dimensional matrix that maps u to y:

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ CB & 0 & 0 & 0 \\ CAB & CB & 0 & 0 \\ CA^2B & CAB & CB & 0 \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}}_{\Phi} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix}$$

In other words, $\|G\|_\infty = \|\Phi\|$. This matrix Φ is block lower-triangular (because the map $u \to y$ is causal), and it has a block-Toeplitz structure (because the map $u \to y$ is time-invariant). This is different from the Hankel matrix we saw when discussing the Hankel singular values of a system (that matrix mapped *past* inputs to *future* outputs, whereas the matrix Φ maps *future* inputs to *future* outputs). Unlike the Hankel case, the matrix Φ does not have a nice factorization. Instead, we will use a Lyapunov-like approach.

Suppose we can find a function $V : \mathbb{R}^n \to \mathbb{R}$ such that

(i) $V(x) > 0 \quad \forall x \neq 0$

(ii) $V(x_{t+1}) - V(x_t) \leq \gamma^2 \|u_t\|^2 - \|y_t\|^2$

Then we can sum the second inequality and use the first inequality to deduce:

$$-V(x_0) \leq V(x_N) - V(x_0) \leq \gamma^2 \sum_{t=0}^{N-1} \|u_t\|^2 - \sum_{t=0}^{N-1} \|y_t\|^2$$

Rearranging, taking the limit $N \to \infty$, and assuming $u \in \ell_2$, we obtain

$$\sum_{t=0}^{N-1} \|y_t\|^2 - V(x_0) \leq \gamma^2 \sum_{t=0}^{N-1} \|u_t\|^2 \leq \gamma^2 \|u\|^2$$

This holds for all N, so we deduce that $y \in \ell_2$ as well, and we have

$$\|y\|^2 - V(x_0) \leq \gamma^2 \|u\|^2$$

This leads to:

$$\frac{\|y\|^2}{\|u\|^2} - \frac{V(x_0)}{\|u\|^2} \leq \gamma^2$$

This holds for all u, so in particular, since the system is linear, we can scale u and y will scale as well. We deduce that

$$\|y\| \leq \gamma \|u\|$$

Let's look for a quadratic $V(x)$. Specifically, pick $V(x) = x^\mathsf{T} P x$, with $P \succ 0$. Substituting this into the second inequality, we obtain:

$$x_{t+1}^\mathsf{T} P x_{t+1} - x_t^\mathsf{T} P x_t \leq \gamma^2 \|u_t\|^2 - \|y_t\|^2$$

Substituting $x_{t+1} = Ax_t + Bu_t$ and $y_t = Cx_t$, we obtain a quadratic inequality in (x_t, u_t), which we can write as

$$\begin{bmatrix} x_t \\ u_t \end{bmatrix}^\mathsf{T} \begin{bmatrix} A^\mathsf{T} P A - P + C^\mathsf{T} C & A^\mathsf{T} P B \\ B^\mathsf{T} P A & B^\mathsf{T} P B - \gamma^2 I \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} \preceq 0$$

this will be satisfied if the matrix in the middle is negative semidefinite. All we have proven so far is that if we can find $P \succ 0$ such that this matrix condition holds, we will have proven that the gain of the system G is less than γ. It turns out the converse is true as well! A quadratic function P with these properties exists *if and only if* the system has gain bound less than γ. This result is known as the *bounded real lemma*. Here is a complete statement.

Theorem 3.1. *Suppose (A, B, C) is a minimal realization. In other words, (A, B) is controllable and (A, C) is observable. Then the following statements are equivalent.*

(i) $\|y\| \leq \gamma \|u\|$ *for all* $u \in \ell_2$

(ii) There exists a matrix $P \succ 0$ *that satisfies*

$$\begin{bmatrix} A^\mathsf{T} P A - P + C^\mathsf{T} C & A^\mathsf{T} P B \\ B^\mathsf{T} P A & B^\mathsf{T} P B - \gamma^2 I \end{bmatrix} \preceq 0 \qquad (2)$$

Eq. (2) in Theorem 3.1 is a linear matrix inequality. The set of P that satisfy it is a convex set, since P appears linearly. Moreover, it is also linear in γ^2. So the optimization problem of finding the smallest γ^2 such that there exists a $P \succ 0$ satisfying (2) is a convex optimization problem.

For a detailed proof of Theorem 3.1, please see the supplementary notes.

3.1 Schur Complements

A useful property of Schur complements is that:

$$\begin{bmatrix} A & B \\ B^\mathsf{T} & D \end{bmatrix} \succ 0 \iff \begin{Bmatrix} A \succ 0 \\ D - B^\mathsf{T} A^{-1} B \succ 0 \end{Bmatrix} \iff \begin{Bmatrix} D \succ 0 \\ A - B D^{-1} B^\mathsf{T} \succ 0 \end{Bmatrix}$$

This result follows from the block-LDU and block-UDL factorizations seen earlier in the class. We can therefore apply this to the LMI (2) and conclude that:

$$\begin{aligned}
\begin{bmatrix} A^\mathsf{T} P A - P + C^\mathsf{T} C & A^\mathsf{T} P B \\ B^\mathsf{T} P A & B^\mathsf{T} P B - \gamma^2 I \end{bmatrix} &\prec 0 \\
\begin{bmatrix} P - A^\mathsf{T} P A & -A^\mathsf{T} P B \\ -B^\mathsf{T} P A & \gamma^2 I - B^\mathsf{T} P B \end{bmatrix} - \begin{bmatrix} C^\mathsf{T} \\ 0 \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix} &\succ 0 \\
\begin{bmatrix} P - A^\mathsf{T} P A & -A^\mathsf{T} P B & C^\mathsf{T} \\ -B^\mathsf{T} P A & \gamma^2 I - B^\mathsf{T} P B & 0 \\ C & 0 & I \end{bmatrix} &\succ 0 \\
\begin{bmatrix} P & 0 & C^\mathsf{T} \\ 0 & \gamma^2 I & 0 \\ C & 0 & I \end{bmatrix} - \begin{bmatrix} A^\mathsf{T} \\ B^\mathsf{T} \\ 0 \end{bmatrix} P \begin{bmatrix} A & B & 0 \end{bmatrix} &\succ 0 \\
\begin{bmatrix} P & 0 & C^\mathsf{T} & A^\mathsf{T} \\ 0 & \gamma^2 I & 0 & B^\mathsf{T} \\ C & 0 & I & 0 \\ A & B & 0 & P^{-1} \end{bmatrix} &\succ 0 \\
\begin{bmatrix} P & 0 & C^\mathsf{T} & A^\mathsf{T} P \\ 0 & \gamma^2 I & 0 & B^\mathsf{T} P \\ C & 0 & I & 0 \\ PA & PB & 0 & P \end{bmatrix} &\succ 0
\end{aligned}$$

We can also make the observation that since $A^\mathsf{T} P A - P + C^\mathsf{T} C$ is negative definite, then $A^\mathsf{T} P A - P + C^\mathsf{T} C + H = 0$ for some $H \succ 0$, and we can conclude from our standard results on Lyapunov equations that A must be Schur-stable. In other words, having a finite system gain implies stability. For linear finite-dimensional systems, the converse is also true (stable systems must have a finite gain bound).