ME 7247: Advanced Control Systems	**Fall 2022–23**

Lecture 15: Balanced Realization

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In the last lecture we saw controllability and observability ellipsoids. In this lecture we want to combine these two concepts to see what states are both easy to control and observe simultaneously. This will lead us to two interesting applications: **System identification** and **Model reduction**.

1 Change of coordinates

A linear discrete-time system can be modeled as

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t \\ \mathbf{y}_t &= \mathbf{C}\mathbf{x}_t \end{aligned} \tag{1}$$

where $\mathbf{x}_t \in \mathbb{R}^n$, $\mathbf{u}_t \in \mathbb{R}^m$ and $\mathbf{y}_t \in \mathbb{R}^p$ are state vector, control input and measured output, respectively. Matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are of compatible sizes. In many practical applications, the state vector cannot be directly observed. In fact, the system model is so complicated (and sometimes even unknown) that we prefer to treat it as a black box, i.e., we give it a set of inputs $\mathbf{u} = \{u_0, u_1, \dots, u_n\}$ and get a set of outputs $\mathbf{y} = \{y_0, y_1, \dots, y_n\}$ and we want to identify the corresponding system, i.e., estimate the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in equation (1). In this section, we show that given a set of inputs and its corresponding set of outputs, one cannot find a *unique* mapping between the input and the output. In other words, the state vector $\mathbf{x}$ or even its size is not unique.

Let $\mathbf{T} \in \mathbb{R}^{n\times q}$ be an invertible matrix such that

$$\mathbf{x}_t = \mathbf{T}\mathbf{z}_t. \tag{2}$$

Substituting this equation in the equation (1) we get:

$$\begin{aligned} \mathbf{z}_{t+1} &= (\overbrace{\mathbf{T}^{-1}\mathbf{A}\mathbf{T}}^{\mathbf{A}_{\text{new}}})\mathbf{z}_t + (\overbrace{\mathbf{T}^{-1}\mathbf{B}}^{\mathbf{B}_{\text{new}}})\mathbf{u}_t \\ \mathbf{y}_t &= (\underbrace{\mathbf{C}\mathbf{T}}_{\mathbf{C}_{\text{new}}})\mathbf{z}_t. \end{aligned} \tag{3}$$

So, we see that we have the exact same input/output mapping, but with a new state vector $\mathbf{z}$ and new system matrices such that

$$(\mathbf{A}, \mathbf{B}, \mathbf{C}) \to (\mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \mathbf{T}^{-1}\mathbf{B}, \mathbf{C}\mathbf{T}) \tag{4}$$

This gives rise to two interesting questions;

- How can we identify a realization of a system (i.e., find $\mathbf{A}, \mathbf{B}, \mathbf{C}$) using the set of inputs and corresponding outputs?

- Given a system, how can we use this mapping to decrease the system's order (i.e., truncate the state vector) to have a less complicated model and vice versa?

In the remaining of this lecture we try to tackle these questions. Before that, we want to investigate the effect of coordinate change on observability, controllability, impulse response and transfer function of a given system.

1.1 Observability and controllability

Recall we defined the controllability matrix as

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots \end{bmatrix} \tag{5}$$

using mapping (4) we can write

$$\mathcal{C}_{\text{new}} = \begin{bmatrix} \mathbf{T}^{-1}\mathbf{B} & \mathbf{T}^{-1}\mathbf{AB} & \mathbf{T}^{-1}\mathbf{A}^2\mathbf{B} & \cdots \end{bmatrix} = \mathbf{T}^{-1}\mathcal{C}.$$

Likewise, for the observability matrix

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \end{bmatrix} \tag{6}$$

so

$$\mathcal{O}_{\text{new}} = \begin{bmatrix} \mathbf{CT} \\ \mathbf{CAT} \\ \mathbf{CA}^2\mathbf{T} \\ \vdots \end{bmatrix} = \mathcal{O}\mathbf{T}.$$

Controllability and observability Gramians were defined as $\mathbf{P} = \mathcal{C}\mathcal{C}^\top$ and $\mathbf{Q} = \mathcal{O}^\top\mathcal{O}$. Using the new observability and controllability matrices we just derived, we have

$$\begin{aligned} \mathbf{P}_{\text{new}} &= \mathcal{C}_{\text{new}}\mathcal{C}_{\text{new}}^\top = \mathbf{T}^{-1}\mathbf{P}\mathbf{T}^{-\top} \\ \mathbf{Q}_{\text{new}} &= \mathcal{O}_{\text{new}}^\top\mathcal{O}_{\text{new}} = \mathbf{T}^\top\mathbf{Q}\mathbf{T}. \end{aligned}$$

Note that the eigenvalues of the new Gramians are not the same as the old ones, therefore, controllability and observability ellipsoids change when mapping the system. However, the rank of these matrices (whether the system is controllable or observable) is not affected by the change of coordinates.

1.2 Impulse response

We set $\mathbf{u}_0 = \mathbf{I}_m$, $\mathbf{x}_0 = \mathbf{0}_n$ and $\mathbf{u}_t = \mathbf{0}_m$ for all $t \geq 1$ and using system equations (1) compute the output

$$\begin{aligned} \mathbf{y}_0 &= \mathbf{C}\mathbf{x}_0 = \mathbf{0}_p \\ \mathbf{y}_1 &= \mathbf{CB} \\ \mathbf{y}_2 &= \mathbf{CAB} \\ \mathbf{y}_3 &= \mathbf{CA}^2\mathbf{B} \\ &\cdots. \end{aligned}$$

We do the same for the system (3) by letting $\mathbf{z}_0 = \mathbf{0}_q$ and the same input. So, we get

$$\begin{aligned}
\mathbf{y}_0 &= \mathbf{C}\mathbf{T}\mathbf{z}_0 = \mathbf{0}_p \\
\mathbf{y}_1 &= \mathbf{C}\mathbf{T}\mathbf{T}^{-1}\mathbf{C} = \mathbf{C}\mathbf{B} \\
\mathbf{y}_2 &= \mathbf{C}\mathbf{A}\mathbf{B} \\
\mathbf{y}_3 &= \mathbf{C}\mathbf{A}^2\mathbf{B} \\
&\dots .
\end{aligned}$$

So, the impulse response, as expected, is not affected by the change of coordinates. The matrices $\mathbf{C}\mathbf{A}^k\mathbf{B}$ for $k = 0, 1, \dots$ are called the *Markov parameters* of the system.

1.3 Transfer function

Applying the shift operator $\mathcal{Z}$ (the equivalent of Laplace transform $\mathcal{L}$) on the equation of the system in (1) we get the transfer function

$$G(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

doing the same for the system in (4) we have

$$G_{\text{new}}(z) = \mathbf{C}\mathbf{T}(z\mathbf{I} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T})^{-1}\mathbf{T}^{-1}\mathbf{B}$$

factoring out a $\mathbf{T}$ from the left side and a $\mathbf{T}^{-1}$ from the right side of the the parenthesis, straightforwardly we get

$$G(z) = G_{\text{new}}(z).$$

So, the transfer function is invariant of the mapping matrix $\mathbf{T}$.

2 Hankel operator

For a given system, we run an experiment, which consists of two steps:

1. Apply inputs $\{\dots, \mathbf{u}_{-3}, \mathbf{u}_{-2}, \mathbf{u}_{-1}\}$ starting from $\mathbf{x}_{-\infty} = 0$ so the system is driven to $\mathbf{x}_0$.
2. At time $t = 0$ stop applying inputs and start measuring the outputs $\{\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \dots\}$.

Now we try to find the input/output map. Using (1) we can write

$$\begin{aligned}
\mathbf{y}_0 &= \mathbf{C}\mathbf{x}_0 \\
&= \mathbf{C}(\mathbf{A}\mathbf{x}_{-1} + \mathbf{B}\mathbf{u}_{-1}) \\
&= \mathbf{C}\mathbf{B}\mathbf{u}_{-1} + \mathbf{C}\mathbf{A}(\mathbf{A}\mathbf{x}_{-2} + \mathbf{B}\mathbf{u}_{-2}) \\
&= \mathbf{C}\mathbf{B}\mathbf{u}_{-1} + \mathbf{C}\mathbf{A}\mathbf{B}\mathbf{u}_{-2} + \mathbf{C}\mathbf{A}^2(\mathbf{A}\mathbf{x}_{-3} + \mathbf{B}\mathbf{u}_{-3}) \\
&\ \ \vdots \\
&= \sum_{i=1}^{\infty} \mathbf{C}\mathbf{A}^{i-1}\mathbf{B}\mathbf{u}_{-i}
\end{aligned}$$

likewise, for $\mathbf{y}_k = \mathbf{C}\mathbf{x}_k = \mathbf{C}A^k\mathbf{x}_0$ we can write

$$\mathbf{y}_k = \sum_{i=1}^{\infty} \mathbf{C}\mathbf{A}^{k+i-1}\mathbf{B}\mathbf{u}_{-i}.$$

As a result, we can write the input/output mapping as

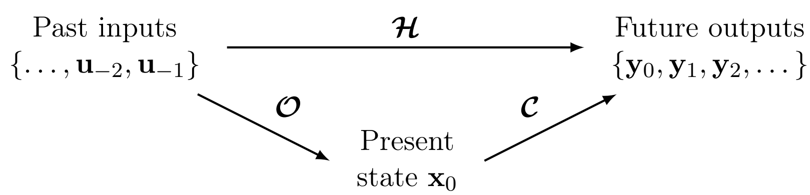
$$\begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{CB} & \mathbf{CAB} & \mathbf{CA}^2\mathbf{B} & \dots \\ \mathbf{CAB} & \mathbf{CA}^2\mathbf{B} & \mathbf{CA}^3\mathbf{B} & \dots \\ \mathbf{CA}^2\mathbf{B} & \mathbf{CA}^3\mathbf{B} & \mathbf{CA}^4\mathbf{B} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\boldsymbol{\mathcal{H}} \triangleq \text{ Hankel operator}} \begin{bmatrix} \mathbf{u}_{-1} \\ \mathbf{u}_{-2} \\ \mathbf{u}_{-3} \\ \vdots \end{bmatrix}. \tag{7}$$

A *Hankel* matrix is a matrix whose anti-diagonal entities are the same (a matrix with the same property but for the diagonal elements is called a *Toeplitz* matrix). Plus, $\boldsymbol{\mathcal{H}}$ is composed of the Markov parameters so is invariant under coordinate changes.

Considering the definitions of observability and controllability matrices in (5) and (6), we can decompose the Hankel matrix as

$$\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{O}}\boldsymbol{\mathcal{C}} \tag{8}$$

which was expected, as the Hankel matrix maps the output to the input, controllability matrix maps the input to the state vector and observability matrix maps the state vector to the output. We can represent this using the diagram:



Using equations (8), (5), (6) and the mapping in (4) it is straightforward to check the Independence of $\boldsymbol{\mathcal{H}}$ from $\mathbf{T}$. From this factorization of $\boldsymbol{\mathcal{H}}$ it is obvious that $\text{rank}(\boldsymbol{\mathcal{H}}) = n$. Two important applications of the Hankel operator are model reduction and system identification, which are to be discussed in what follows.

3 Balanced realization

Sometimes there are some state variables in a system that are hard or even impossible to either control or observe. For example, when there is an unstable element in the system (e.g., a rotating disk with an ever-increasing angular velocity), if we are not able to control or observe it, there is no point in including it in the state vector. The problem that we are trying to solve here, is to detect those states and, in the next step, remove them from the state vector.

As we saw in the previous lecture, from the controllability ellipsoid of a system we could detect states that are easily controllable (in the direction of the major axis of the ellipsoid) and vice versa.

We had the same property for the observability ellipsoid, with the most observable state in the direction of the major axis. In general, these two ellipsoids are totally different, i.e., the states that are easy to control might be hard to observe! It is desirable to find the state variables that are both easily controllable and observable and give them higher priority compared to other states. Even we can discard some states that are costly or hard to observe or/and control. As we saw in Section 1.1, the controllability and observability Gramians transform if we change coordinates.

It turns out that a change of coordinates $\mathbf{T}$ exists that makes the controllability and observability Gramians match, and become diagonal! In other words,

$$\mathbf{P}_{\text{new}} = \mathbf{Q}_{\text{new}} = \mathbf{\Sigma}. \tag{9}$$

We choose $\mathbf{\Sigma}$ to be the matrix of singular values of the Hankel matrix of the system. If we look at the eigenvalues of $\mathbf{PQ}$, we have

$$\lambda(\mathbf{PQ}) = \lambda(\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{C}}^\top\boldsymbol{\mathcal{O}}^\top\boldsymbol{\mathcal{O}}) = \lambda((\boldsymbol{\mathcal{O}}\boldsymbol{\mathcal{C}})(\boldsymbol{\mathcal{O}}\boldsymbol{\mathcal{O}})^\top) = \lambda(\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^\top) = \sigma(\boldsymbol{\mathcal{H}})^2. \tag{10}$$

So, the eigenvalues of $\mathbf{PQ}$ are squared singular values of the Hankel matrix

$$\sigma(\boldsymbol{\mathcal{H}}) = \sqrt{\lambda(\mathbf{PQ})}.$$

Additionally, we know that the eigenvalues of $\mathbf{PQ}$ and $\mathbf{P}^{1/2}\mathbf{Q}\mathbf{P}^{1/2}$ are the same and equal to their singular values, as these matrices are symmetric and positive definite. So, as we saw in equation (10) the singular values of $\mathbf{P}^{1/2}\mathbf{Q}\mathbf{P}^{1/2}$ are squared singular values of the Hankel matrix. As a result, if $\mathbf{\Sigma}$ is the diagonal matrix containing singular values of the Hankel matrix, we can write the SVD of $\mathbf{P}^{1/2}\mathbf{Q}\mathbf{P}^{1/2}$ as

$$\mathbf{P}^{1/2}\mathbf{Q}\mathbf{P}^{1/2} = \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}^\top \tag{11}$$

multiplying by $\mathbf{\Sigma}^{-1/2}\mathbf{U}^\top(\dots)\mathbf{U}\mathbf{\Sigma}^{-1/2}$, we obtain

$$\mathbf{\Sigma}^{-1/2}\mathbf{U}^\top\mathbf{P}^{1/2}\mathbf{Q}\mathbf{P}^{1/2}\mathbf{U}\mathbf{\Sigma}^{-1/2} = \mathbf{\Sigma}. \tag{12}$$

As discussed in Section 1.1, we have

$$\mathbf{Q}_{\text{new}} = \mathbf{T}^\top\mathbf{Q}\mathbf{T}.$$

Comparing this equation with equations (9) and (12) we can define

$$\mathbf{T} \triangleq \mathbf{P}^{1/2}\mathbf{U}\mathbf{\Sigma}^{-1/2}. \tag{13}$$

This definition of $\mathbf{T}$ is based on the fact that $\mathbf{Q}_{\text{new}} = \mathbf{\Sigma}$. For $\mathbf{P}_{\text{new}}$ we also have

$$\mathbf{P}_{\text{new}} = \mathbf{T}^{-1}\mathbf{P}\mathbf{T}^{-\top} = \mathbf{\Sigma}^{1/2}\mathbf{U}^\top\mathbf{P}^{-1/2}\mathbf{P}\mathbf{P}^{-1/2}\mathbf{U}\mathbf{\Sigma}^{1/2} = \mathbf{\Sigma}$$

This checks out! So, we have found a transformation $\mathbf{T}$ that makes the Gramians equal and diagonal. Since $\mathbf{\Sigma}$ is diagonal, the resulting ellipsoids will always be aligned with the axes such that z_1 is the most controllable and observable state and z_q is the least controllable and observable state.

To sum up, for any given system with $\mathbf{x}$ as its state vector, and $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ matrices, we can find a new state vector $\mathbf{z} = \mathbf{T}^{-1}\mathbf{x}$ with new matrices $\mathbf{A}_{\text{new}}, \mathbf{B}_{\text{new}}$ and $\mathbf{C}_{\text{new}}$ such that the controllability and observability ellipsoids match and in the new state vector states are placed in accordance with their controllability and observability properties, with the most controllable and observable state coming first. The Matlab command to get a balanced realization of a system is `balreal`.

Balanced truncation. After obtaining a balanced realization of a given system, we can perform model reduction. There are several ways to do that, including:

- **Petrov—Galerkin truncation.** The easiest way of model reduction is to discard k last states. This method is called Petrov—Galerkin truncation approach. However, the point here is that the states are coupled, i.e., in general $\mathbf{A}_{\text{new}}$ is not diagonal, so it is not the best practice to just remove the k last states. Predictably, the truncated system will not match the steady-state response of the original system. However, it will do a good job matching the transient response.

- **DC matching.** If we want our reduced system's steady-state response to match that of the original system, we can use this method. There are several methods to do DC matching. Although this method preserve the steady-state response of the original system, there is no guarantee that it will also match the transient response. The Matlab command for this family of methods is

```
balred(SYS,ORDER) % calculates an ORDER order approximation of the system
   using DC matching method
```

A numerical example. Consider the following system of the form (1)

$$\mathbf{x}_{t+1} = \begin{bmatrix} 0.5 & -0.1 \\ 0.4 & -0.1 \end{bmatrix} \mathbf{x}_t + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \mathbf{u}_t \tag{14a}$$

$$\mathbf{y}_t = \begin{bmatrix} 4 & 0 \end{bmatrix} \mathbf{x}_t. \tag{14b}$$

We can find Gramians:

```
P = dlyap(A,B*B')
Q = dlyap(A',C'*C)

P =
    1.0499    3.0276
    3.0276    9.0159
Q =
   20.8600   -0.9639
   -0.9639    0.1912
```

Now, we can plot controllability and observability ellipsoids; see Fig. 1. Obviously, the states that are easy to control are hard to observe and vice versa. We can calculate the balanced realization:

```
    [U,S2,~] = svd(sqrtm(P)*Q*sqrtm(P));
    T = sqrtm(P)*U*inv(sqrtm(sqrtm(S2)));
    Ab = T\A*T;
    Bb = T\B;
    Cb = C*T;
```

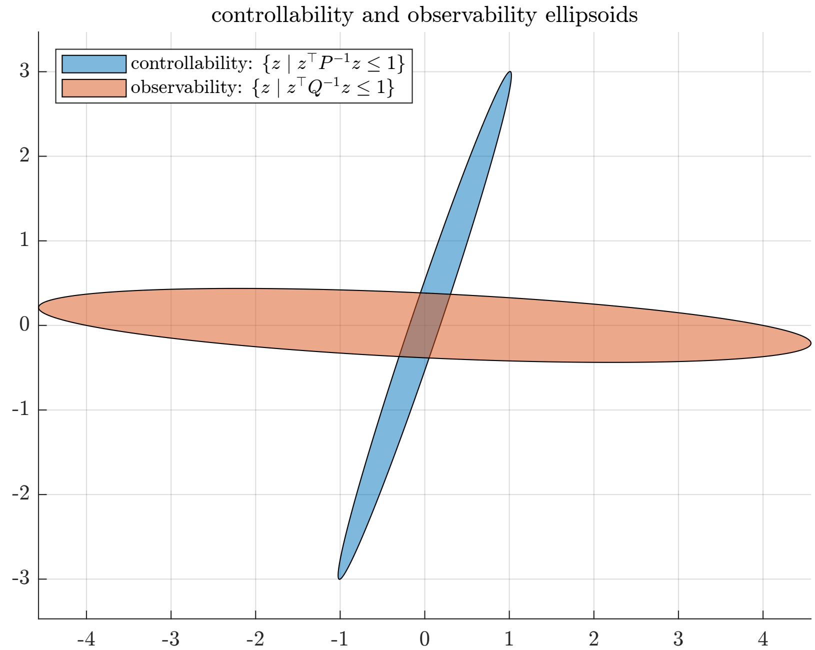


Figure 1: Controllability and observability ellipsoids for the example of Eq. (14). The directions that are easy to control are also (roughly) difficult to observe, and vice versa.

The balanced realization of the system becomes

$$\mathbf{x}_{t+1} = \begin{bmatrix} 0.2245 & 0.2223 \\ 0.2223 & 0.1755 \end{bmatrix} \mathbf{x}_t + \begin{bmatrix} -1.997 \\ 0.1097 \end{bmatrix} \mathbf{u}_t \tag{15a}$$

$$\mathbf{y}_t = \begin{bmatrix} -10.997 & 0.1097 \end{bmatrix} \mathbf{x}_t \tag{15b}$$

for which we can compute new Gramians

```
Pb = dlyap(Ab, Bb*Bb');
Qb = dlyap(Ab', Cb'*Cb);
```

which results in

$$\mathbf{P}_{\text{new}} = \begin{bmatrix} 4.2114 & 0 \\ 0 & 0.2271 \end{bmatrix}$$

$$\mathbf{Q}_{\text{new}} = \begin{bmatrix} 4.2114 & 0 \\ 0 & 0.2271 \end{bmatrix}.$$

We can see that the new Gramians are the same. We can also plot the new controllability and observability ellipsoids.

Finally, we can perform a truncation to see how the reduced system (with only one state) compares to the original system (with two states). We used both methods presented in previous sections: truncation and singular perturbation (DC match). The first has better agreement for initial transients, while the second has better steady-state agreement. Here is a plot comparing the performance of the original system and both 1st order approximations.

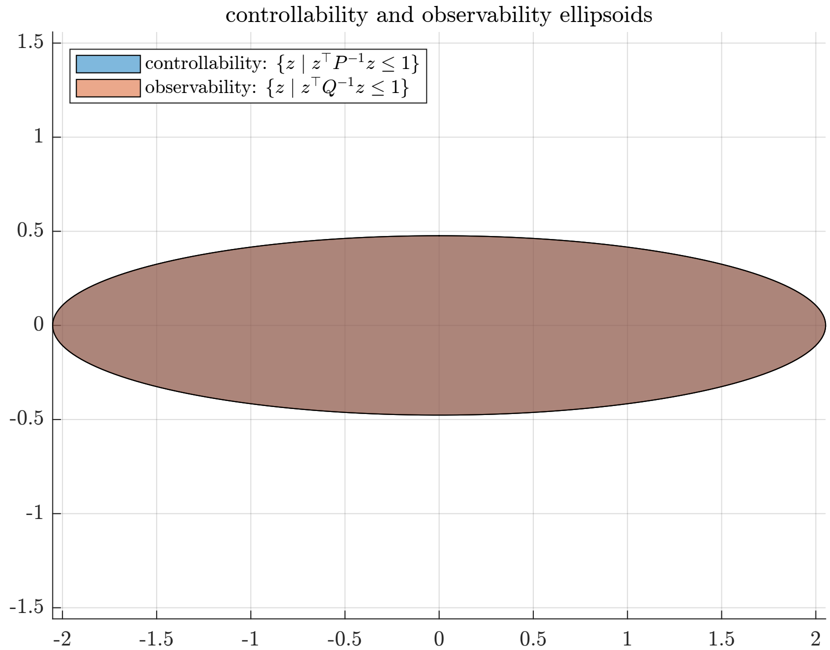


Figure 2: Controllability and observability ellipsoids for the example of Eq. (15). This is a balanced realization so controllability and observability ellipsoids coincide perfectly and are aligned with the axes. The easiest direction to control and observe is x_1.

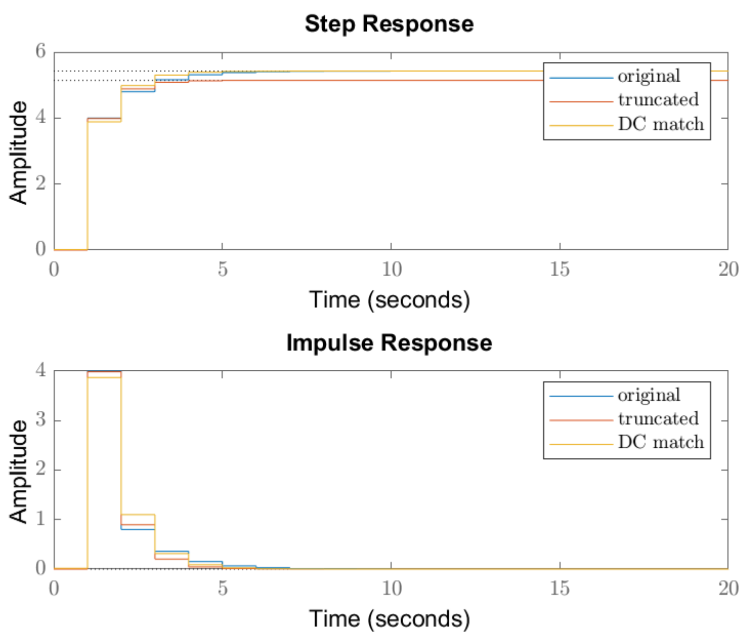


Figure 3: Step and impulse responses of the system Eq. (14) using different model reduction approaches.

4 System identification

The second application of Hankel operator is system identification. In this application, we do not know the system, instead we have a set of inputs and its corresponding set of outputs. There are several methods of identifying a system. One of the most famous algorithms is called **Ho-Kalman** method. Comparing the Hankel matrix from Eq. (7) to the Markov parameters of the system with an impulse input, we observe that the first column of the Hankel matrix is actually the sequence of the outputs. The second column is the first column shifted up one step. As a result, we have

$$\hat{\boldsymbol{\mathcal{H}}} = \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 & \iddots \\ \mathbf{y}_2 & \mathbf{y}_3 & \iddots & \\ \mathbf{y}_3 & \iddots & & \\ \iddots & & & \end{bmatrix}. \tag{16}$$

Now we get to choose the size of the state vector. To this end, we compute the SVD decomposition of $\hat{\boldsymbol{\mathcal{H}}}$ and keep the most significant singular values. We can discard those close to zero. Now, we have the dimension of the state vector. Using (8) we decompose the Hankel matrix to controllability and observability matrices. This decomposition is not unique, so we can choose any decomposition with compatible matrix sizes. Now, we have $\hat{\boldsymbol{\mathcal{O}}}$ and $\hat{\boldsymbol{\mathcal{C}}}$. The first block of the controllability matrix is $\mathbf{B}$ (see (5)) and the first block of the observability matrix is $\mathbf{C}$ (see (6)), so, from $\hat{\boldsymbol{\mathcal{O}}}$ and $\hat{\boldsymbol{\mathcal{C}}}$ we have $\hat{\mathbf{C}}$ and $\hat{\mathbf{B}}$. To compute $\hat{\mathbf{A}}$ we form the shifted Hankel matrix. To do that, we remove the first measurement from the original Hankel matrix in (16)

$$\hat{\boldsymbol{\mathcal{H}}}^{\uparrow} = \begin{bmatrix} \mathbf{y}_2 & \mathbf{y}_3 & \mathbf{y}_4 & \iddots \\ \mathbf{y}_3 & \mathbf{y}_4 & \iddots & \\ \mathbf{y}_4 & \iddots & & \\ \iddots & & & \end{bmatrix}. \tag{17}$$

The relation between the shifted Hankel matrix and controllability and observability matrices is given by

$$\hat{\boldsymbol{\mathcal{H}}}^{\uparrow} = \boldsymbol{\mathcal{O}}\mathbf{A}\boldsymbol{\mathcal{C}}.$$

So, we can obtain an estimate of $\mathbf{A}$ using the following equation

$$\hat{\mathbf{A}} = \hat{\boldsymbol{\mathcal{O}}}^{\dagger}\hat{\boldsymbol{\mathcal{H}}}^{\uparrow}\hat{\boldsymbol{\mathcal{C}}}^{\dagger}. \tag{18}$$

The three matrices $\hat{\mathbf{A}}, \hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ represent an n-dimensional realization of the system obtained entirely from the input-output data.