ME 7247: Advanced Control Systems **Fall 2021–22**

Lecture 12: Steady-state Kalman filter

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This lecture, we focus on the convergence of Kalman Filter. As mentioned in the previous lecture, the Kalman Filter can be used to estimate the system state x. The tracking capability that using $\hat{x}$ to estimate x in a dynamical system is also needed to consider. In the lecture, we show that the Kalman Filter will always converge to some certain *steady state* no matter the stability of the original dynamical system.

1 Kalman Filter Framework

Following from the previous lecture, consider a dynamical system with state space equations

$$x_{t+1} = Ax_t + w_t \qquad \text{(process/state equation)} \tag{1}$$
$$y_t = Cx_t + v_t \qquad \text{(measurement equation)} \tag{2}$$

in which the initial condition and noises are assumed to be normally distributed such that

$$\begin{aligned} x_0 &\sim \mathcal{N}(\mu_0, \Sigma_0) \\ w_t &\sim \mathcal{N}(0, W) \\ v_t &\sim \mathcal{N}(0, V). \end{aligned}$$

Additionally, the noises w_t and v_t are independent of $x_0, x_1, \dots, x_t$, and the $w_0, \dots, w_t, v_0, \dots, v_t$ are mutually independent.

Since we have no direct access to the actual state x_t, the goal will be estimating x_t based on the measurements $y_0, y_1, \dots, y_{t-1}$. To obtain the estimates,

$$\begin{aligned} \hat{x}_t &= \mathbf{E}(x_t \mid y_0, y_1, \dots, y_{t-1}) \\ \Sigma_t &= \mathbf{Cov}(x_t \mid y_0, y_1, \dots, y_{t-1}), \end{aligned}$$

we can use the Kalman Filter:

$$\boxed{\begin{aligned} L_t &= -A\Sigma_t C^\mathsf{T}\left(C\Sigma_t C^\mathsf{T} + V\right)^{-1} \\ \hat{x}_{t+1} &= (A + L_t C)\hat{x}_t - L_t y_t \\ \Sigma_{t+1} &= (A + L_t C)\Sigma_t A^\mathsf{T} + W \end{aligned}}$$

where L_t is also known as the *Kalman gain.*

Note: The Kalman filter is also a common state observer used for linear systems. In other parts of the world (in Sweden, for example), people use K for the Kalman Filter gain and L for the controller gain. In the US, we typically write K for the controller gain and L for the observer gain since L is also for the *Luenberger Observer.*

Overall, the structure of a system with a Kalman Filter is shown in Fig. 1.

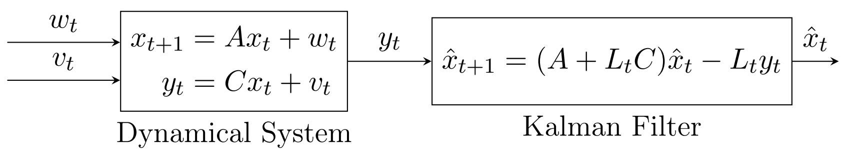

Figure 1: Block diagram of the system with Kalman Filter. w_t and v_t could be disturbances or noises. y_t is the measurement of the dynamical system. $\hat{x}_t$ is the online estimate of system state x_t.

2 Convergence without Measurement Update

Let us first consider the case that there is no measurement update. Then, all we have is just the system propagating on its own through time and we are trying to keep track of the estimates. The estimator will look like

$$\hat{x}_{t+1} = A\hat{x}_t \tag{3}$$

$$\boxed{\Sigma_{t+1} = A\Sigma_t A^\mathsf{T} + W} \tag{4}$$

in which the estimate mean and covariances only update based on the latest previous estimate. What we really care about in this case is whether or not the Σ_t converges as $t \to \infty$.

Intuitively, we can start the discussion of the convergence problem in a noiseless scalar example:

$$x_{t+1} = Ax_t + w_t$$

where A, x_t, w_t are assumed to be scalars. First, consider the there is no noise, i.e. $w_t = 0$. If A has a magnitude greater than one, then x_t will go to infinity as $t \to \infty$. If A has a magnitude less than one, then x_t will converge to zero as $t \to \infty$. Whereas, in the case we have a Gaussian distributed noise $w_t \sim \mathcal{N}(0, W)$, the state x_t will never converge to zero due to the randomness added from the noise. Similar to the noiseless case, if A has a magnitude less than one, instead of converges to zero, x_t will reach some steady state that the amount of decrease caused by the multiplication of A is balanced by the amount of increase caused by adding the noise. At the end, the randomness of x_t as $t \to \infty$ should reach a steady state, which can be described by a steady-state covariance. So, if the covariance Σ_t converges as $t \to \infty$ and the limit is assumed to be

$$\lim_{t\to\infty} \Sigma_t = \Sigma$$

then according to Eq. (4), in the steady state, we should have

$$\boxed{\Sigma = A\Sigma A^\mathsf{T} + W} \tag{5}$$

which is in the form of *discrete Lyapunov equation* and can be solved in MATLAB with command `Sigma = dlyap(A,W)`. Theorem 2.1 provides the conditions of the convergence.

Note: The continuous Lyapunov equation for a continuous system $\dot{x} = Ax$ should be in the form of

$$A\Sigma + \Sigma A^\mathsf{T} + W = 0$$

which can be solved in MATLAB with command `Sigma = lyap(A,W)`.

Theorem 2.1. *Let $W \succ 0$, the followings are equivalent:*

(i) For any Σ_0, $\Sigma_{t+1} = A\Sigma_t A^\mathsf{T} + W$ converges to Σ. (Independent of Σ_0)

(ii) The Lyapunov equation $A\Sigma A^\mathsf{T} - \Sigma + W = 0$ has a unique solution and $\Sigma \succ 0$.

(iii) The matrix A is Schur-stable.

Note: The spectral radius $\rho(A)$ is the largest eigenvalue magnitude among all eigenvalues of A. We say that A is *Schur-stable* (or just *stable*) when $\rho(A) < 1$. Equivalently, $\lim_{t\to\infty} A^t = 0$.

Brief proof of (i). To prove (i) based on either of (ii) and (iii), we can subtract Eq. (5) from Eq. (4)

$$\Sigma_{t+1} - \Sigma = A\big(\Sigma_t - \Sigma\big)A^\mathsf{T}$$

and by iterating, we obtain

$$\Sigma_t - \Sigma = A^t\big(\Sigma_0 - \Sigma\big)\big(A^\mathsf{T}\big)^t.$$

If A is Schur-stable, then $A^t \to 0$ as $t \to \infty$, so we have $\Sigma_t \to \Sigma$ as $t \to \infty$. ■

Brief proof of (iii). To prove (iii) based on (i) and (ii), suppose $A^\mathsf{T} v = \lambda v$. Rewrite Eq. (4) as

$$v^*\Sigma_{t+1}v = |\lambda|^2 v^*\Sigma_t v + v^*Wv \tag{6}$$

where v^* is the conjugate transpose of v (note that λ and v are complex, in general). Let $v_{t+1} = v^*\Sigma_{t+1}v$ and $c = v^*Wv$, Eq. (6) becomes a scalar equation

$$v_{t+1} = |\lambda|^2 v_t + c$$

where c is a positive number since $W \succ 0$. Therefore, v_t is

$$v_t = |\lambda|^{2t} + \big(1 + |\lambda|^2 + |\lambda|^4 + \cdots + |\lambda|^{2t-2}\big)\, c$$

and the geometric series of v_t only converges when $|\lambda| < 1$. And in this case,

$$\lim_{t\to\infty} v_t = \frac{c}{1 - |\lambda|^2}.$$

■

Difference between eigenvalues and singular values in terms of system stability

In the previous lectures, we talked about singular values whereas we only discuss the eigenvalues in this lecture. Although in some cases singular values and eigenvalues are the same thing, there are some differences in terms of the stability of A. For example, in our case, we have the system

$$x_{t+1} = Ax_t.$$

All eigenvalues of A have magnitude less than one would be both necessary and sufficient for the system to be stable, i.e.

$$|\lambda(A)| < 1 \iff \|x_t\| \to 0 \text{ as } t \to \infty.$$

As for the singular values, the singular values of A are less than one is sufficient for the system to be stable, i.e.

$$\sigma_1 < 1 \Longrightarrow \|x_t\| \to 0 \text{ as } t \to \infty.$$

The difference can be checked by comparing the spectral radius $\rho(A)$ of A and the the largest singular values. It is a fact that

$$\rho(A) := \max_i |\lambda_i(A)| \leq \sigma_1.$$

It is possible to have $\sigma_1 > 1$ and $\rho(A) < 1$ at the same time such that the system is still stable. An example could be the case that

$$A = \begin{bmatrix} 0.9 & 0 \\ 100 & 0.9 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where A has singular values of 100.0081 and 0.0081 and eigenvalues of 0.9 and 0.9. The norm of x_t can be calculated as

$$\|x_t\| = \|A^t x_0\|.$$

For the first 200 iterations, the norm $\|x_t\|$ is shown in Fig. 2 where $\|x_t\|$ does not decrease to zero monotonically while the system is still stable.

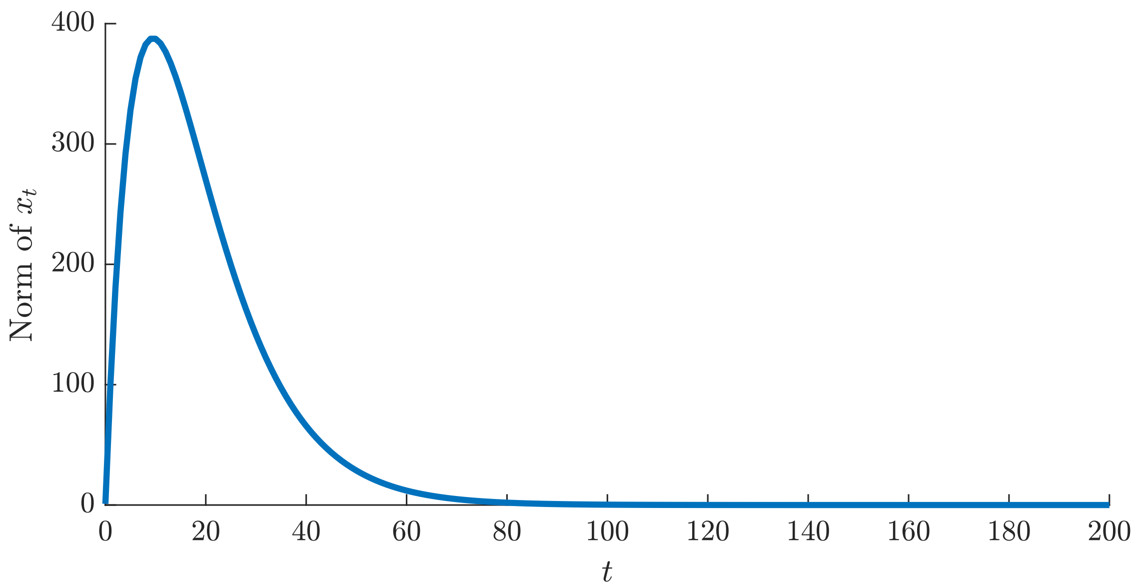

Figure 2: Example of a stable system $x_{t+1} = Ax_t$ where A has singular values of 100.0081 and 0.0081.

For the case that the singular values of A are less than one, we can also write

$$\sigma_1 < 1 \Longleftrightarrow \|x_t\| \to 0 \text{ monotonically}, \quad \text{as } t \to \infty.$$

which can be proved with the definition of matrix norms

$$\|x_{t+1}\| = \|Ax_t\| \leq \|A\|\|x_t\| = \sigma_1\|x_t\|,$$

where

$$\|A\| = \max \frac{\|Ax\|}{\|x\|} = \sigma_1.$$

3 Convergence with Measurement Update

In Section 2, without the measurement update, the convergence of the covariance in Eq. (4) dependent on the stability of A. Similarly, for the version with measurement update, we can also find out in what cases the covariance of the estimates converges to a certain "value". In this case, we have the Kalman filter with initial conditions:

$$\begin{aligned}\hat{x}_0 &= \mu_0 \\ \Sigma_0 &= \Sigma_0 \\ L_t &= -A\Sigma_t C^\mathsf{T}\big(C\Sigma_t C^\mathsf{T} + V\big)^{-1} \\ \hat{x}_{t+1} &= (A + L_t C)\hat{x}_t - L_t y_t \\ \Sigma_{t+1} &= A\Sigma_t A^\mathsf{T} + W - A\Sigma_t C^\mathsf{T}\big(C\Sigma_t C^\mathsf{T} + V\big)^{-1} C\Sigma_t A^\mathsf{T} \\ &= (A + L_t C)\Sigma_t A^\mathsf{T} + W\end{aligned} \tag{7}$$

We would like for the error covariance Σ_t to converge to some Σ, which would mean that the Kalman gain L_t will also converge to some L. In practice, the Kalman filter is often run for a long time, so to good approximation, we can just compute Σ and L and use the *steady-state filter*

$$\hat{x}_{t+1} = (A + LC)\hat{x}_t - Ly_t. \tag{8}$$

Similar to Eq. (4) and Eq. (5), if Σ_t in Eq. (7) converges to Σ, then Σ must satisfy the discrete time algebraic Riccati equation (ARE):

$$\boxed{A\Sigma A^\mathsf{T} - \Sigma + W - A\Sigma C^\mathsf{T}\big(C\Sigma C^\mathsf{T} + V\big)^{-1} C\Sigma A^\mathsf{T} = 0}. \tag{9}$$

Theorem 3.1. *If $W \succeq 0$, $V \succ 0$, (C, A) is detectable, and (A, W) is stabilizable, then:*

(i) The ARE (9) has a unique solution satisfying $\Sigma \succ 0$. This is called the "stabilizing solution".

(ii) Σ can be found by iterating Eq. (7) starting from any initial $\Sigma_0 \succeq 0$.

(iii) Let $L := -A\Sigma C^\mathsf{T}\big(C\Sigma C^\mathsf{T} + V\big)^{-1}$. The matrix $A + LC$ is Schur-stable.

Note:

- (C, A) detectable means that there exists an L such that $A + LC$ is stable. When A is stable, we can just pick $L = 0$ so this is always true.
- (A, W) stabilizable means there exists a K such that $A + WK$ is stable. When $W \succ 0$, we can just pick $K = -W^{-1}A$, so this is always true.
- For (i), Eq. (9) has many other solutions, but there is only one solution satisfying $\Sigma \succ 0$.
- The discrete time algebraic Riccati equation Eq. (9) can be solved in MATLAB with command `idare`. Similarly, the continuous time algebraic Riccati equation can be solved in MATLAB with command `icare`.

- The ARE Eq. (9) can also be rewritten in the form of Lyapunov equation:

$$\big(A + LC\big)\Sigma\big(A + LC\big)^{\mathsf{T}} - \Sigma + \big(W + LVL^{\mathsf{T}}\big) = 0 \tag{10}$$

  which is important when analyzing the error dynamics.

- In the scalar case, Lyapunov equations are linear equations and algebraic Riccati equations are quadratic equations. So you can think about these matrix equations as generalizations of their scalar counterparts.

4 Error Dynamics

By substituting the measurement Eq. (2) into the steady-state Kalman Filter Eq. (8), we can rewrite the state updating equation as

$$\hat{x}_{t+1} = (A + LC)\hat{x}_t - L\big(Cx_t + v_t\big). \tag{11}$$

Then, defining the tracking error as $e_t := x_t - \hat{x}_t$ and subtracting Eq. (11) from Eq. (1)

$$x_{t+1} = Ax_t + w_t,$$

we have the error dynamics equation

$$e_{t+1} = \big(A + LC\big)e_t + \big(w_t + Lv_t\big) \tag{12}$$

which does not directly depend on the measurements. In Eq. (12), $w_t + Lv_t$ is a combination of noises w_t and v_t. Additionally, from Theorem 3.1 we have that $A + LC$ is stable, therefore, the error will eventually reach a steady state. To find the steady-state covariance, we can take the covariance of both sides of Eq. (12), and in the limit $t \to \infty$, we recover the Lyapunov equation Eq. (10).

Thus, supported by Theorem 2.1 and Theorem 3.1, the error dynamics Eq. (12) should always be stable no matter the stability of the original system A.

These comments apply for any other Luenberger observer as well. For example, suppose we pick any other L (not necessarily the Kalman gain) such that $A + LC$ is stable. Then, if we define the observer dynamics

$$\hat{x}_{t+1} = (A + LC)\hat{x}_t - Ly_t$$

as before, then we will obtain the same error dynamics of Eq. (12) and we can evaluate the steady-state error covariance associated with this new L by solving the Lyapunov equation Eq. (10).

5 Spring-mass-damper example revisited

Recall the spring-mass-damper example from Lecture 11. We will revisit this example in light of the steady-state results derived in this lecture.

First, we can check that the error covariance tends to a steady state. In Fig. 3, we plot the diagonal components of Σ_t with no measurements (time update only) and with Kalman filtering. We also plot the components of L_t on a shorter timescale. Two observations:

- The no-measurement covariances are larger than the KF covariances. This is because the diagonal components we are plotting are the variances of the individual states, and those will always improve when measurements are observed.
- The no-measurement covariances take longer to converge than the KF covariances. This is because the no-measurement covariance convergence depends on the eigenvalues of A while the KF covariance convergence depends on the eigenvalues of $A+LC$. We have $\rho(A+LC) < \rho(A)$ so the KF gains converge faster.

In Fig. 4, we run the same code on an unstable version of the system (changed the signs of the dampers b_i). In this case, the error dynamics are still stable but the no-measurement covariance is unstable because $\rho(A) > 1$.

In the unstable case, even though the state x_t is unbounded, the error $x_t - \hat{x}_t$ is stable, so our estimate tracks the true state well.

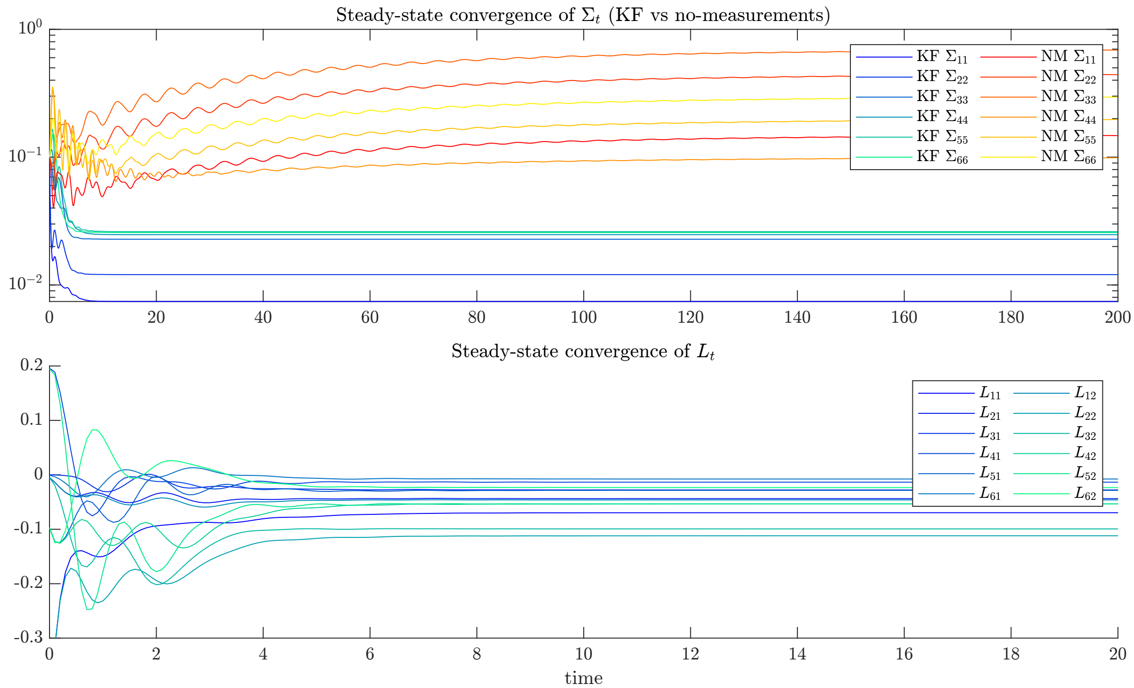

Figure 3: Long-term trends for Σ_t (time update only vs. Kalman Filter) and the Kalman gain L_t. Both converge, but the KF gains converge faster. (note the different time axes)

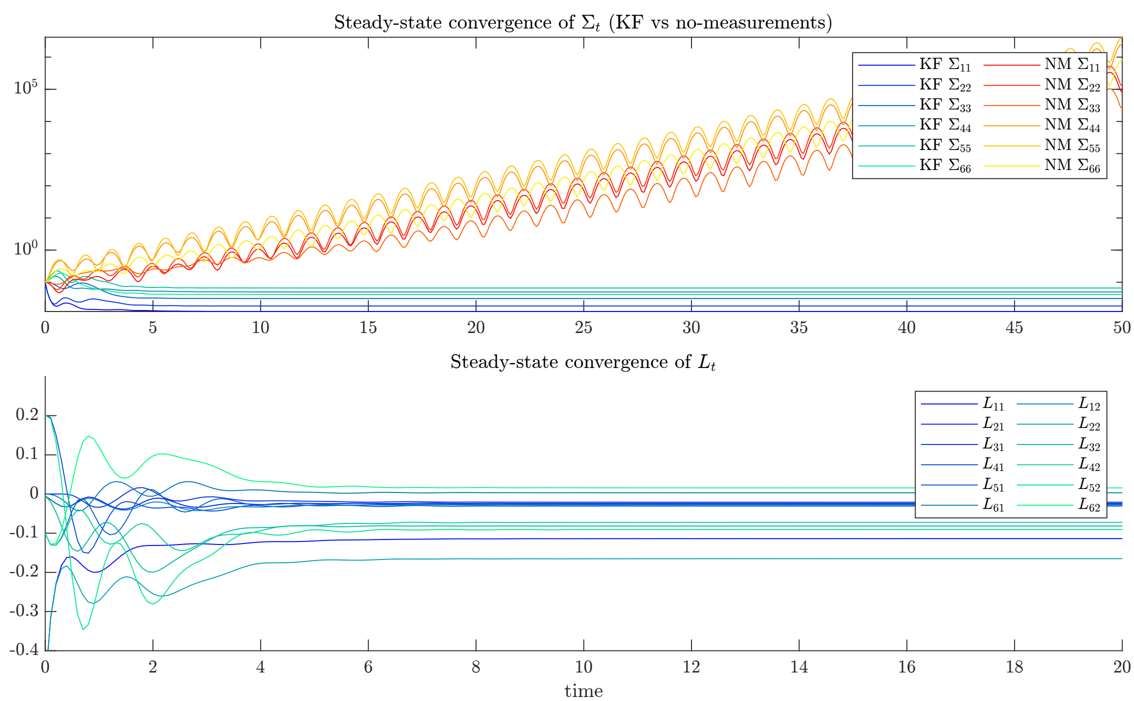

Figure 4: Same as Fig. 3, but with unstable open-loop dynamics. The KF error is still stable, but the no-measurement error grows without bound.