

6. Duality

- Estimating LP bounds
- LP duality
- Simple example
- Sensitivity and shadow prices
- Complementary slackness
- Optimality conditions

The Top Brass example revisited

$$\begin{array}{ll} \text{maximize}_{f,s} & 12f + 9s \\ \text{subject to:} & 4f + 2s \leq 4800, \quad f + s \leq 1750 \\ & 0 \leq f \leq 1000, \quad 0 \leq s \leq 1500 \end{array}$$

Suppose the maximum profit is p^* . How can we **bound** p^* ?

- Finding a *lower* bound is easy... pick any feasible point!
 - ▶ $\{f = 0, s = 0\}$ is feasible. So $p^* \geq 0$ (we can do better...)
 - ▶ $\{f = 500, s = 1000\}$ is feasible. So $p^* \geq 15000$.
 - ▶ $\{f = 1000, s = 400\}$ is feasible. So $p^* \geq 15600$.
- Each feasible point of the LP yields a lower bound for p^* .
- Finding the largest lower bound = solving the LP!

Estimating upper bounds

$$\begin{array}{ll} \text{maximize}_{f,s} & 12f + 9s \\ \text{subject to:} & 4f + 2s \leq 4800, \quad f + s \leq 1750 \\ & 0 \leq f \leq 1000, \quad 0 \leq s \leq 1500 \end{array}$$

Suppose the maximum profit is p^* . How can we **bound** p^* ?

- Finding an *upper* bound is harder... (use the constraints!)
 - ▶ $12f + 9s \leq 12 \cdot 1000 + 9 \cdot 1500 = 25500$. So $p^* \leq 25500$.
 - ▶ $12f + 9s \leq f + (4f + 2s) + 7(f + s)$
 $\leq 1000 + 4800 + 7 \cdot 1750 = 18050$. So $p^* \leq 18050$.
- Combining the constraints in different ways yields different upper bounds on the optimal profit p^* .

Estimating upper bounds

$$\begin{array}{ll} \text{maximize}_{f,s} & 12f + 9s \\ \text{subject to:} & 4f + 2s \leq 4800, \quad f + s \leq 1750 \\ & 0 \leq f \leq 1000, \quad 0 \leq s \leq 1500 \end{array}$$

Suppose the maximum profit is p^* . How can we **bound** p^* ?

What is the **best** upper bound we can find by combining constraints in this manner?

Estimating upper bounds

$$\begin{array}{ll} \text{maximize} & 12f + 9s \\ & \text{subject to:} \\ & 4f + 2s \leq 4800, \quad f + s \leq 1750 \\ & 0 \leq f \leq 1000, \quad 0 \leq s \leq 1500 \end{array}$$

- Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ be the multipliers. If we can choose them such that for *any* feasible f and s , we have:

$$12f + 9s \leq \lambda_1(4f + 2s) + \lambda_2(f + s) + \lambda_3f + \lambda_4s \quad (1)$$

Then, using the constraints, we will have the following upper bound on the optimal profit:

$$12f + 9s \leq 4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4$$

Estimating upper bounds

$$\begin{array}{ll} \text{maximize}_{f,s} & 12f + 9s \\ \text{subject to:} & 4f + 2s \leq 4800, \quad f + s \leq 1750 \\ & 0 \leq f \leq 1000, \quad 0 \leq s \leq 1500 \end{array}$$

- Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ be the multipliers. If we can choose them such that for *any* feasible f and s , we have:

$$12f + 9s \leq \lambda_1(4f + 2s) + \lambda_2(f + s) + \lambda_3f + \lambda_4s \quad (1)$$

Rearranging (1), we get:

$$0 \leq (4\lambda_1 + \lambda_2 + \lambda_3 - 12)f + (2\lambda_1 + \lambda_2 + \lambda_4 - 9)s$$

We can ensure this always holds by choosing $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ to make the bracketed terms nonnegative.

Estimating upper bounds

$$\begin{array}{ll} \text{maximize}_{f,s} & 12f + 9s \\ \text{subject to:} & 4f + 2s \leq 4800, \quad f + s \leq 1750 \\ & 0 \leq f \leq 1000, \quad 0 \leq s \leq 1500 \end{array}$$

- **Recap:** If we choose $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ such that:

$$4\lambda_1 + \lambda_2 + \lambda_3 \geq 12 \quad \text{and} \quad 2\lambda_1 + \lambda_2 + \lambda_4 \geq 9$$

Then we have a *upper* bound on the optimal profit:

$$p^* \leq 4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4$$

Finding the best (smallest) upper bound is... an LP!

The dual of Top Brass

$$\begin{array}{ll} \text{maximize} & 12f + 9s \\ & f, s \\ \text{subject to:} & 4f + 2s \leq 4800, \quad f + s \leq 1750 \\ & 0 \leq f \leq 1000, \quad 0 \leq s \leq 1500 \end{array}$$

To find the best upper bound, solve the **dual** problem:

$$\begin{array}{ll} \text{minimize} & 4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \\ \text{subject to:} & 4\lambda_1 + \lambda_2 + \lambda_3 \geq 12 \\ & 2\lambda_1 + \lambda_2 + \lambda_4 \geq 9 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{array}$$

The dual of Top Brass

Primal problem:

$$\begin{array}{ll} \text{maximize}_{f,s} & 12f + 9s \\ \text{subject to:} & 4f + 2s \leq 4800 \\ & f + s \leq 1750 \\ & f \leq 1000 \\ & s \leq 1500 \\ & f, s \geq 0 \end{array}$$

Solution is p^* .

Dual problem:

$$\begin{array}{ll} \text{minimize}_{\lambda_1, \dots, \lambda_4} & 4800\lambda_1 + 1750\lambda_2 \\ & + 1000\lambda_3 + 1500\lambda_4 \\ \text{subject to:} & 4\lambda_1 + \lambda_2 + \lambda_3 \geq 12 \\ & 2\lambda_1 + \lambda_2 + \lambda_4 \geq 9 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{array}$$

Solution is d^* .

- Primal is a maximization, dual is a minimization.
- There is a dual variable for each primal constraint.
- There is a dual constraint for each primal variable.
- (any feasible primal point) $\leq p^* \leq d^* \leq$ (any feasible dual point)

The dual of Top Brass

Primal problem:

$$\begin{aligned} \max_{f,s} \quad & \begin{bmatrix} 12 \\ 9 \end{bmatrix}^T \begin{bmatrix} f \\ s \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} 4 & 2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f \\ s \end{bmatrix} \leq \begin{bmatrix} 4800 \\ 1750 \\ 1000 \\ 1500 \end{bmatrix} \\ & f, s \geq 0 \end{aligned}$$

Dual problem:

$$\begin{aligned} \min_{\lambda_1, \dots, \lambda_4} \quad & \begin{bmatrix} 4800 \\ 1750 \\ 1000 \\ 1500 \end{bmatrix}^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} 4 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \geq \begin{bmatrix} 12 \\ 9 \end{bmatrix} \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{aligned}$$

Using matrix notation...

Code: [Top Brass dual.ipynb](#)

General LP duality

Primal problem (P)

$$\begin{array}{ll} \underset{x}{\text{maximize}} & c^T x \\ \text{subject to:} & Ax \leq b \\ & x \geq 0 \end{array}$$

Dual problem (D)

$$\begin{array}{ll} \underset{\lambda}{\text{minimize}} & b^T \lambda \\ \text{subject to:} & A^T \lambda \geq c \\ & \lambda \geq 0 \end{array}$$

If x and λ are feasible points of (P) and (D) respectively:

$$c^T x \leq p^* \leq d^* \leq b^T \lambda$$

Powerful fact: if p^* and d^* exist and are finite, then $p^* = d^*$.
This property is known as **strong duality**.

General LP duality

Primal problem (P)

$$\begin{array}{ll} \underset{x}{\text{maximize}} & c^T x \\ \text{subject to:} & Ax \leq b \\ & x \geq 0 \end{array}$$

1. optimal p^* is attained
2. unbounded: $p^* = +\infty$
3. infeasible: $p^* = -\infty$

Dual problem (D)

$$\begin{array}{ll} \underset{\lambda}{\text{minimize}} & b^T \lambda \\ \text{subject to:} & A^T \lambda \geq c \\ & \lambda \geq 0 \end{array}$$

1. optimal d^* is attained
2. unbounded: $d^* = -\infty$
3. infeasible: $d^* = +\infty$

Which combinations are possible? Remember: $p^* \leq d^*$.

General LP duality

Primal problem (P)

$$\begin{array}{ll} \underset{x}{\text{maximize}} & c^T x \\ \text{subject to:} & Ax \leq b \\ & x \geq 0 \end{array}$$

Dual problem (D)

$$\begin{array}{ll} \underset{\lambda}{\text{minimize}} & b^T \lambda \\ \text{subject to:} & A^T \lambda \geq c \\ & \lambda \geq 0 \end{array}$$

There are **exactly four** possibilities:

1. (P) and (D) are both feasible and bounded, and $p^* = d^*$.
2. $p^* = +\infty$ (unbounded primal) and $d^* = +\infty$ (infeasible dual).
3. $p^* = -\infty$ (infeasible primal) and $d^* = -\infty$ (unbounded dual).
4. $p^* = -\infty$ (infeasible primal) and $d^* = +\infty$ (infeasible dual).

Which case is it?

$$\begin{array}{ll} \text{maximize} & 3x_1 + 3x_2 \\ & x_1, x_2 \geq 0 \\ \text{such that:} & x_1 + x_2 \leq 2 \\ & x_1 + x_2 \leq 2 \end{array}$$

(feasible and bounded)

$$\begin{array}{ll} \text{minimize} & 2\lambda_1 + 2\lambda_2 \\ & \lambda_1, \lambda_2 \geq 0 \\ \text{such that:} & \lambda_1 + \lambda_2 \geq 3 \\ & \lambda_1 + \lambda_2 \geq 3 \end{array}$$

(feasible and bounded)

$$\begin{array}{ll} \text{maximize} & 3x_1 + 3x_2 \\ & x_1, x_2 \geq 0 \\ \text{such that:} & -x_1 - x_2 \leq 2 \\ & -x_1 - x_2 \leq 2 \end{array}$$

(feasible and unbounded)

$$\begin{array}{ll} \text{minimize} & 2\lambda_1 + 2\lambda_2 \\ & \lambda_1, \lambda_2 \geq 0 \\ \text{such that:} & -\lambda_1 - \lambda_2 \geq 3 \\ & -\lambda_1 - \lambda_2 \geq 3 \end{array}$$

(infeasible)

Which case is it?

$$\begin{array}{ll} \text{maximize} & -3x_1 - 3x_2 \\ & x_1, x_2 \geq 0 \\ \text{such that:} & x_1 + x_2 \leq 2 \\ & x_1 + x_2 \leq 2 \end{array}$$

(feasible and bounded)

$$\begin{array}{ll} \text{minimize} & 2\lambda_1 + 2\lambda_2 \\ & \lambda_1, \lambda_2 \geq 0 \\ \text{such that:} & \lambda_1 + \lambda_2 \geq -3 \\ & \lambda_1 + \lambda_2 \geq -3 \end{array}$$

(feasible and bounded)

$$\begin{array}{ll} \text{maximize} & 3x_1 + 3x_2 \\ & x_1, x_2 \geq 0 \\ \text{such that:} & x_1 - x_2 \leq -2 \\ & -x_1 + x_2 \leq -2 \end{array}$$

(infeasible)

$$\begin{array}{ll} \text{minimize} & -2\lambda_1 - 2\lambda_2 \\ & \lambda_1, \lambda_2 \geq 0 \\ \text{such that:} & \lambda_1 - \lambda_2 \geq 3 \\ & -\lambda_1 + \lambda_2 \geq 3 \end{array}$$

(infeasible)

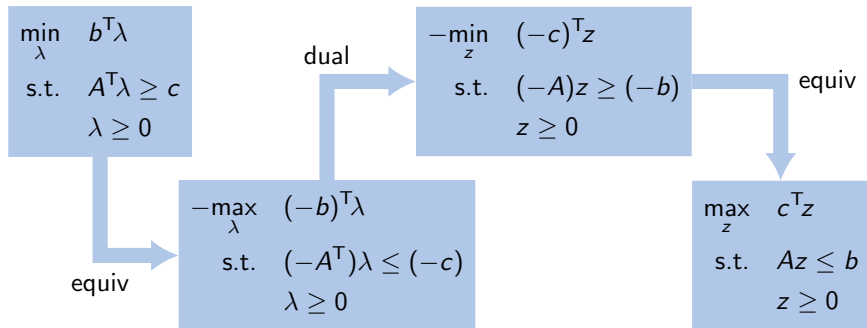
More properties of the dual

To find the dual of an LP that is **not** in standard form:

1. convert the LP to standard form
2. write the dual
3. make simplifications

True for LP duality,
not true in general.

Example: What is the dual of the dual? **the primal!**



More duals

Standard form:

$$\begin{array}{ll} \max_x & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array}$$

dual

$$\begin{array}{ll} \min_{\lambda} & b^T \lambda \\ \text{s.t.} & \lambda \geq 0 \\ & A^T \lambda \geq c \end{array}$$

Free form:

$$\begin{array}{ll} \max_x & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \text{ free} \end{array}$$

dual

$$\begin{array}{ll} \min_{\lambda} & b^T \lambda \\ \text{s.t.} & \lambda \geq 0 \\ & A^T \lambda = c \end{array}$$

Mixed constraints:

$$\begin{array}{ll} \max_x & c^T x \\ \text{s.t.} & Ax \leq b \\ & Fx = g \\ & x \text{ free} \end{array}$$

dual

$$\begin{array}{ll} \min_{\lambda, \mu} & b^T \lambda + g^T \mu \\ \text{s.t.} & \lambda \geq 0 \\ & \mu \text{ free} \\ & A^T \lambda + F^T \mu = c \end{array}$$

More duals

Equivalences between primal and dual problems

Minimization	Maximization
Nonnegative variable \geq	Inequality constraint \leq
Nonpositive variable \leq	Inequality constraint \geq
Free variable	Equality constraint $=$
Inequality constraint \geq	Nonnegative variable \geq
Inequality constraint \leq	Nonpositive variable \leq
Equality constraint $=$	Free Variable

Simple example

Why should we care about the dual?

1. It can sometimes make a problem easier to solve

$\begin{array}{ll} \max_{x,y,z} & 3x + y + 2z \\ \text{s.t.} & x + 2y + z \leq 2 \\ & x, y, z \geq 0 \end{array}$	dual \longleftrightarrow	$\begin{array}{ll} \min_{\lambda} & 2\lambda \\ \text{s.t.} & \lambda \geq 3 \\ & 2\lambda \geq 1 \\ & \lambda \geq 2 \\ & \lambda \geq 0 \end{array}$
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- ▶ Dual is much easier in this case!
 - ▶ Many solvers take advantage of duality.
2. Duality is related to the idea of sensitivity: how much do each of your constraints affect the optimal cost?

Sensitivity

Primal problem:

$$\begin{array}{ll} \text{maximize}_{f,s} & 12f + 9s \\ \text{subject to:} & 4f + 2s \leq 4800 \\ & f + s \leq 1750 \\ & f \leq 1000 \\ & s \leq 1500 \\ & f, s \geq 0 \end{array}$$

Solution is p^* .

Dual problem:

$$\begin{array}{ll} \text{minimize}_{\lambda_1, \dots, \lambda_4} & 4800\lambda_1 + 1750\lambda_2 \\ & + 1000\lambda_3 + 1500\lambda_4 \\ \text{subject to:} & 4\lambda_1 + \lambda_2 + \lambda_3 \geq 12 \\ & 2\lambda_1 + \lambda_2 + \lambda_4 \geq 9 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{array}$$

Solution is d^* .

If Millco offers to sell me more **wood** at a price of \$1 per board foot, should I accept the offer?

Sensitivity

Primal problem:

$$\begin{array}{ll} \text{maximize}_{f,s} & 12f + 9s \\ \text{subject to:} & 4f + 2s \leq 4800 \\ & f + s \leq 1750 \\ & f \leq 1000 \\ & s \leq 1500 \\ & f, s \geq 0 \end{array}$$

Solution is p^* .

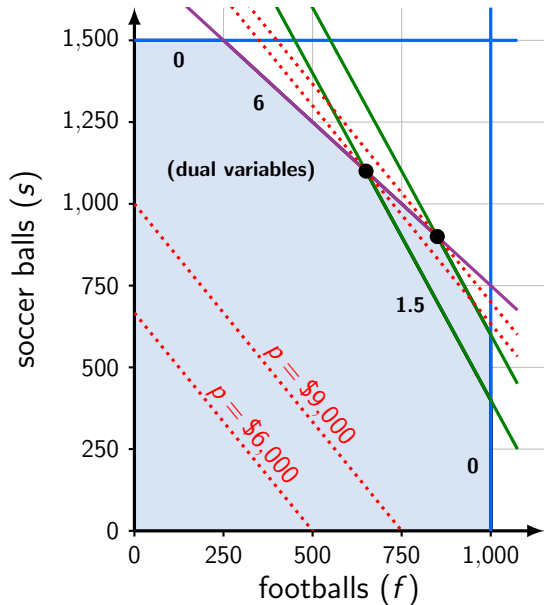
Dual problem:

$$\begin{array}{ll} \text{minimize}_{\lambda_1, \dots, \lambda_4} & 4800\lambda_1 + 1750\lambda_2 \\ & + 1000\lambda_3 + 1500\lambda_4 \\ \text{subject to:} & 4\lambda_1 + \lambda_2 + \lambda_3 \geq 12 \\ & 2\lambda_1 + \lambda_2 + \lambda_4 \geq 9 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{array}$$

Solution is d^* .

- changes in primal *constraints* are changes in the dual *cost*.
- a small change to the feasible set of the primal problem can change the optimal f and s , but $\lambda_1, \dots, \lambda_4$ will not change!
- if we increase **4800** by 1, then $p^* = d^*$ increases by λ_1 .

Sensitivity of Top Brass



$$\begin{aligned} \max_{f,s} \quad & 12f + 9s \\ \text{s.t.} \quad & 4f + 2s \leq 5200 \\ & f + s \leq 1750 \\ & 0 \leq f \leq 1000 \\ & 0 \leq s \leq 1500 \end{aligned}$$

What happens if we add 400 wood?

Profit goes up by \$600!
shadow price is \$1.50,
so \$1 is a good price.

Units

- In Top Brass, the primal variables f and s are the number of football and soccer trophies. The total profit is:

$$\begin{aligned}(\text{profit in \$}) = & \left(12 \frac{\text{\$}}{\text{football trophy}}\right)(f \text{ football trophies}) \\ & + \left(9 \frac{\text{\$}}{\text{soccer trophy}}\right)(s \text{ soccer trophies})\end{aligned}$$

- The dual variables also have units. To find them, look at the cost function for the dual problem:

$$\begin{aligned}(\text{profit in \$}) = & (4800 \text{ board feet of wood})\left(\lambda_1 \frac{\text{\$}}{\text{board feet of wood}}\right) \\ & + (1750 \text{ plaques})\left(\lambda_2 \frac{\text{\$}}{\text{plaque}}\right) + \dots\end{aligned}$$

λ_i is the price that item i is worth to us.

Sensitivity in general

Primal problem (P)

$$\begin{array}{ll} \underset{x}{\text{maximize}} & c^T x \\ \text{subject to:} & Ax \leq b \\ & x \geq 0 \end{array}$$

Dual problem (D)

$$\begin{array}{ll} \underset{\lambda}{\text{minimize}} & b^T \lambda \\ \text{subject to:} & A^T \lambda \geq c \\ & \lambda \geq 0 \end{array}$$

Think of the optimal cost as a function $p^*(b)$. Let's perturb b a tiny bit and think about what happens to p^* .

- A small change in b will change the feasible set of (P). So the optimal x^* (and therefore p^*) may change.
- But the dual problem has the same solution as the primal! And b only appears in the dual objective. So if the change in b is small enough, the optimal λ^* does not change.

Sensitivity in general

Primal problem (P)

$$\begin{array}{ll} \underset{x}{\text{maximize}} & c^T x \\ \text{subject to:} & Ax \leq b \\ & x \geq 0 \end{array}$$

Dual problem (D)

$$\begin{array}{ll} \underset{\lambda}{\text{minimize}} & b^T \lambda \\ \text{subject to:} & A^T \lambda \geq c \\ & \lambda \geq 0 \end{array}$$

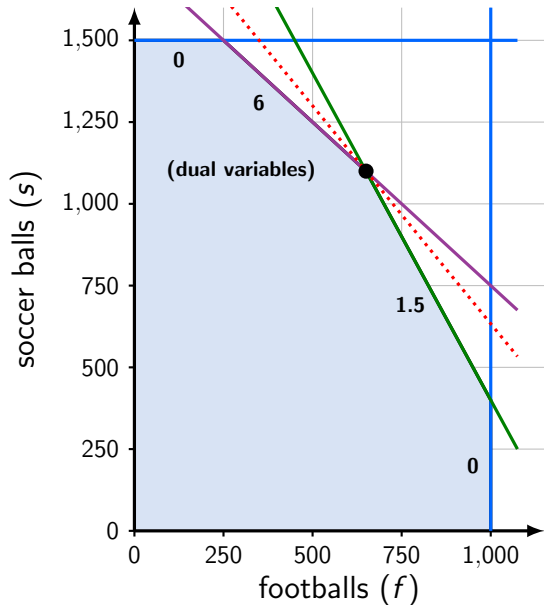
Think of the optimal cost as a function $p^*(b)$. Let's perturb b a tiny bit and think about what happens to p^* .

$$p^*(b) = b_1 \lambda_1^* + \cdots + b_m \lambda_m^*$$

$$\frac{\partial p^*}{\partial b_i} = \lambda_i^*$$

More compactly: $\nabla_b p^* = \lambda^*$.

Sensitivity of Top Brass



$$\begin{aligned} \max_{f, s} \quad & 12f + 9s \\ \text{s.t.} \quad & 4f + 2s \leq 4800 \\ & f + s \leq 1750 \\ & 0 \leq f \leq 1000 \\ & 0 \leq s \leq 1500 \end{aligned}$$

Constraints that are slack at optimality have corresponding dual variables that are zero; those items aren't *worth* anything.

Complementary slackness

- At the optimal point, some inequality constraints become *tight*. Ex: wood and plaque constraints in Top Brass.
- Some inequality constraints may have slack, even at optimality. Ex: brass football/soccer ball constraints.

Either a primal constraint is tight **or** its dual variable is zero.

The same thing happens when we solve the dual problem. Some dual constraints may have slack and others may not.

Either a dual constraint is tight **or** its primal variable is zero.

These properties are called **complementary slackness**.

Proof of complementary slackness

- **Primal:** $\max_x c^T x$ s.t. $Ax \leq b, x \geq 0$
- **Dual:** $\min_\lambda b^T \lambda$ s.t. $A^T \lambda \geq c, \lambda \geq 0$

Suppose (x, λ) is feasible for the primal and the dual.

- Because $Ax \leq b$ and $\lambda \geq 0$, we have: $\lambda^T Ax \leq b^T \lambda$.
- Because $c \leq A^T \lambda$ and $x \geq 0$, we have: $c^T x \leq \lambda^T Ax$.

Combining both inequalities: $c^T x \leq \lambda^T Ax \leq b^T \lambda$.

By strong duality, $c^T x^* = \lambda^{*T} Ax^* = b^T \lambda^*$

Proof of complementary slackness

$$c^T x^* = \lambda^{*T} A x^* = b^T \lambda^*$$

$u_i v_i = 0$ means
that: $u_i = 0$, or
 $v_i = 0$, or *both*.

The first equation says: $x^{*T}(A^T \lambda^* - c) = 0$.
But $x^* \geq 0$ and $A^T \lambda^* \geq c$, therefore:

$$\sum_{i=1}^n x_i^* (A^T \lambda^* - c)_i = 0 \quad \implies \quad x_i^* (A^T \lambda^* - c)_i = 0 \quad \forall i$$

Similarly, the second equation says: $\lambda^{*T}(A x^* - b) = 0$.
But $\lambda^* \geq 0$ and $A x^* \leq b$, therefore:

$$\sum_{j=1}^m \lambda_j^* (A x^* - b)_j = 0 \quad \implies \quad \lambda_j^* (A x^* - b)_j = 0 \quad \forall j$$

Optimality conditions

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to:} & Ax \leq b \\ & x \geq 0 \end{array}$$

Question: If we have a candidate solution x , how can we tell if we have found the best one? i.e., how can we guarantee that x is optimal?

Optimality conditions

$$\begin{array}{ll} \underset{x}{\text{maximize}} & c^T x \\ \text{subject to:} & Ax \leq b \\ & x \geq 0 \end{array}$$

Theorem: The following statements are equivalent:

1. x is a solution of the LP above.
2. There exists a vector λ satisfying:
 - ▶ Primal feasibility: $Ax \leq b$ and $x \geq 0$.
 - ▶ Dual feasibility: $A^T \lambda \geq c$ and $\lambda \geq 0$.
 - ▶ Complementary slackness: $\lambda^T (Ax - b) = 0$.

If such a λ exists, then λ solves the dual LP.

A simple example

Primal problem:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & x_1 + x_2 \\ \text{subject to:} & 2x_1 + x_2 \geq 5 \\ & x_1 + 4x_2 \geq 6 \\ & x_1 \geq 1 \end{array}$$

Dual problem:

$$\begin{array}{ll} \underset{\lambda}{\text{maximize}} & 5\lambda_1 + 6\lambda_2 + \lambda_3 \\ \text{subject to:} & 2\lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1 + 4\lambda_2 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array}$$

Question: Is the point $(x_1, x_2) = (1, 3)$ optimal?

First, we check that it is indeed feasible. Next, we try to find a dual-feasible point that satisfies complimentary slackness.

- Second primal constraint is slack, therefore $\lambda_2 = 0$.
- By dual feasibility, $\lambda_1 + 4\lambda_2 = 1$ and $2\lambda_1 + \lambda_2 + \lambda_3 = 1$.
- Only choice: $\lambda_1 = 1$ and $\lambda_3 = -1$. But $\lambda_3 \not\geq 0$!

$(1, 3)$ is **not optimal** for the primal problem.

A simple example

Primal problem:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & x_1 + x_2 \\ \text{subject to:} & 2x_1 + x_2 \geq 5 \\ & x_1 + 4x_2 \geq 6 \\ & x_1 \geq 1 \end{array}$$

Dual problem:

$$\begin{array}{ll} \underset{\lambda}{\text{maximize}} & 5\lambda_1 + 6\lambda_2 + \lambda_3 \\ \text{subject to:} & 2\lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1 + 4\lambda_2 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array}$$

Another question: Is $(x_1, x_2) = (2, 1)$ optimal?

- It is primal feasible; now let's find λ .
- Third primal constraint is slack, therefore $\lambda_3 = 0$.
- From dual feasibility, $2\lambda_1 + \lambda_2 = 1$ and $\lambda_1 + 4\lambda_2 = 1$.
- Solution is $\lambda_1 = \frac{3}{7}$, $\lambda_2 = \frac{1}{7}$, $\lambda_3 = 0$, which satisfies $\lambda \geq 0$.

$(2, 1)$ is **optimal** for the primal problem.