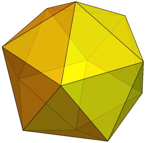
4. Minimax and planning problems

- Optimizing piecewise linear functions
- Minimax problems
- Example: Chebyshev center
- Multi-period planning problems
- Example: building a house

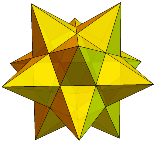
Laurent Lessard (www.laurentlessard.com)

LPs and polyhedra

Linear programs have polyhedral feasible sets:

$\{x \mid Ax \leq b\}$ $\quad\Longrightarrow\quad$ 

Can every polyhedron be expressed as $Ax \leq b$?

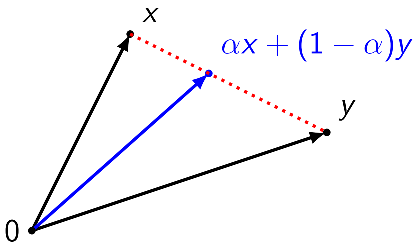
Not this one... 

LPs and polyhedra

If $x, y \in \mathbb{R}^n$, then the linear combination

$$w = \alpha x + (1 - \alpha) y \quad \text{for some } 0 \leq \alpha \leq 1$$

is called a **convex combination**. As we vary α, it traces out the line segment that connects x and y.



LPs and polyhedra

If $Ax \le b$ and $Ay \le b$, and w is a convex combination of x and y, then $Aw \le b$.

Proof: Suppose $w = \alpha x + (1-\alpha)y$.

$$\begin{aligned} Aw &= A\left(\alpha x + (1-\alpha)y\right) \\ &= \alpha Ax + (1-\alpha)Ay \\ &\le \alpha b + (1-\alpha)b \\ &= b \end{aligned}$$

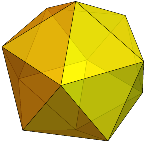
Therefore, $Aw \le b$, which is what we were trying to prove.

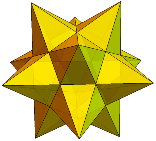
Question: where did we use the fact that $0 \le \alpha \le 1$?

LPs and polyhedra

The previous result implies that every polyhedron describable as $Ax \leq b$ must contain all convex combinations of its points.

- Such polyhedra are called **convex**.
- Informal definition: if you were to "shrink-wrap" it, the entire polyhedron would be covered with no extra space.

Convex: 

Not convex: 

Goes the other way too: every convex polyhedron can be represented as $Ax \leq b$ for appropriately chosen A and b.

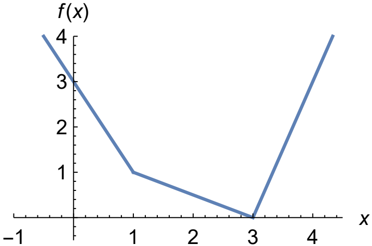
Piecewise linear functions

- Some problems do not appear to be LPs but can be converted to LPs using a suitable transformation.
- An important case: **convex piecewise linear functions**.

Consider the following **nonlinear** optimization:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to:} \quad & x \geq 0 \end{aligned}$$

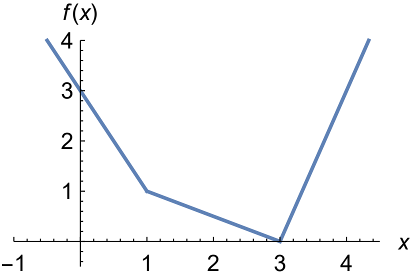
Where $f(x)$ is the function:

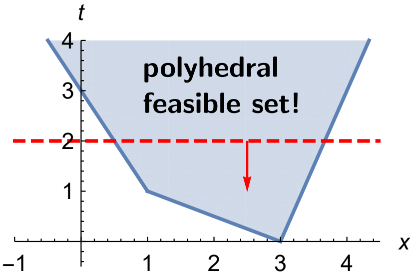


Piecewise linear functions

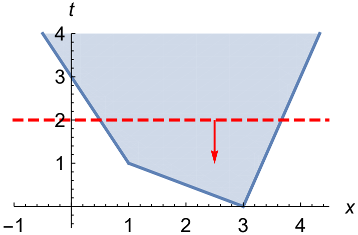
The trick is to convert the problem into **epigraph** form: add an extra decision variable t and turn the cost into a constraint!

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to:} \quad & x \geq 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \underset{x,t}{\text{minimize}} \quad & t \\ \text{subject to:} \quad & t \geq f(x) \\ & x \geq 0 \end{aligned}$$





Piecewise linear functions



$$\begin{array}{ll} \underset{x,t}{\text{minimize}} & t \\ \text{subject to:} & t \ge f(x) \\ & x \ge 0 \end{array}$$

This feasible set is **polyhedral**. It is the set of (x, t) satisfying:

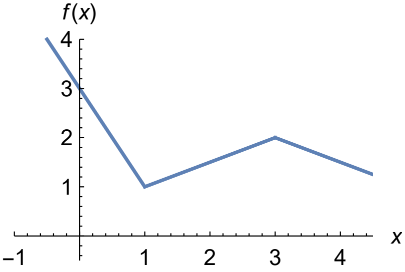
$$\left\{ t \ge -2x + 3, \quad t \ge -\tfrac{1}{2}x + \tfrac{3}{2}, \quad t \ge 3x - 9 \right\}$$

Equivalent linear program:

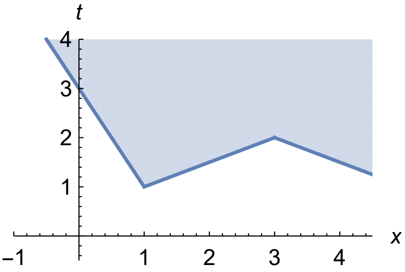
$$\begin{array}{lll} \underset{x,t}{\text{minimize}} & t & \\ \text{subject to:} & t \ge -2x + 3, & t \ge -\tfrac{1}{2}x + \tfrac{3}{2} \\ & t \ge 3x - 9, & x \ge 0 \end{array}$$

Piecewise linear functions

Epigraph trick only works if it's a **convex polyhedron**.



$\Longrightarrow$

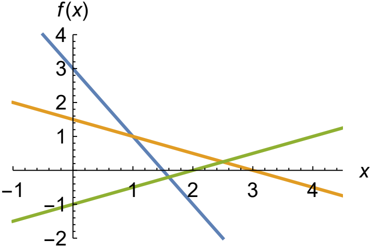


This epigraph is **not a convex polyhedron** so it cannot be the feasible set of a linear program.

Minimax problems

- The maximum of several linear functions is *always* convex. So we can minimize it using the epigraph trick. Example:

$$f(x) = \max_{i=1,\dots,k} \left\{ a_i^\mathsf{T} x + b_i \right\}$$

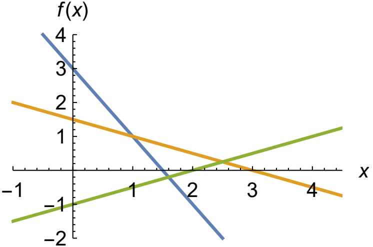


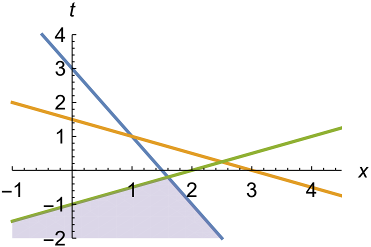
$$\min_x \max_{i=1,\dots,k} \left\{ a_i^\mathsf{T} x + b_i \right\} \quad \Longrightarrow \quad \begin{aligned} \min_{x,t} \quad & t \\ \text{s.t.} \quad & t \ge a_i^\mathsf{T} x + b_i \quad \forall i \end{aligned}$$

Maximin problems

- The minimum of several linear functions is *always* concave. So we can maximize it using the epigraph trick. Example:

$$f(x) = \min_{i=1,\dots,k} \left\{ a_i^\mathsf{T} x + b_i \right\}$$





$$\max_{x} \min_{i=1,\dots,k} \left\{ a_i^\mathsf{T} x + b_i \right\} \quad \Longrightarrow \quad \begin{aligned} \max_{x,t} \quad & t \\ \text{s.t.} \quad & t \le a_i^\mathsf{T} x + b_i \quad \forall i \end{aligned}$$

Minimax and Maximin problems

- A minimax problem:

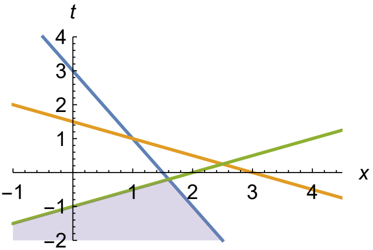
$$\min_{x} \max_{i=1,\ldots,k} \left\{ a_i^T x + b_i \right\} \quad \Longrightarrow \quad \begin{array}{ll} \min\limits_{x,t} & t \\ \text{s.t.} & t \geq a_i^T x + b_i \quad \forall i \end{array}$$

- A maximin problem:

$$\max_{x} \min_{i=1,\ldots,k} \left\{ a_i^T x + b_i \right\} \quad \Longrightarrow \quad \begin{array}{ll} \max\limits_{x,t} & t \\ \text{s.t.} & t \leq a_i^T x + b_i \quad \forall i \end{array}$$

Note: Sometimes called *minmax*, *min-max*, *min/max*.
Of course, minmax $\neq$ maxmin!

Max-Min inequality



In general,

$$\max_x \min_{i=1,\ldots,k} \left\{ a_i^T x + b_i \right\} \quad \leq \quad \min_x \max_{i=1,\ldots,k} \left\{ a_i^T x + b_i \right\}$$

Minimax and Maximin problems

Practical scenario:

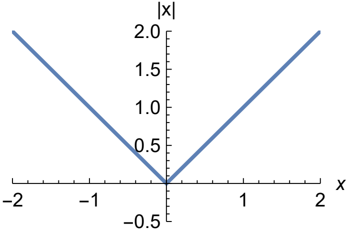
- Paintco produces specialized paints and we are planning production for the coming year. They have some flexibility in how they produce the paints, but ultimately they require employees, as well as electricity, water, and certain chemicals.
- Nobody knows for sure how much paints will sell for, and the future price of electricity, water, and the chemicals is also unknown. But planning decision must be made now.
- Three consulting firms are hired to forecast the costs for the coming year. The three firms return with three different forecasts (cost functions f_1, f_2, f_3). Which one should be used?
- The risk-averse approach is to solve the minimax problem:

$$\min_{x} \max_{i=1,2,3} \{f_i(x)\}$$

Absolute values

Fundamental property:

$$|x| = \max\{x, -x\}$$



Absolute values are piecewise linear.

Absolute values

- Absolute values are piecewise linear!

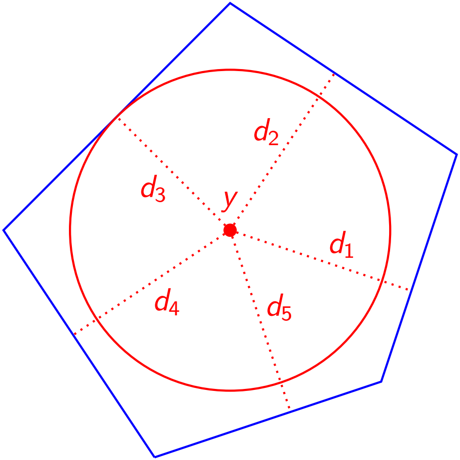
$$\begin{aligned} \min_{x} \quad & |x| \\ \text{s.t.} \quad & Ax \le b \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \min_{x,t} \quad & t \\ \text{s.t.} \quad & Ax \le b \\ & t \ge x \\ & t \ge -x \end{aligned}$$

- So are sums of absolute values:

$$\min_{x,y} \quad |x| + |y| \quad \Longrightarrow \quad \begin{aligned} \min_{x,y,t,r} \quad & t + r \\ \text{s.t.} \quad & t \ge x, \quad t \ge -x \\ & r \ge y, \quad r \ge -y \end{aligned}$$

Chebyshev center

What is the largest sphere you can fit inside a polyhedron?



If y is the center, then draw perpendicular lines to each face of the polyhedron.

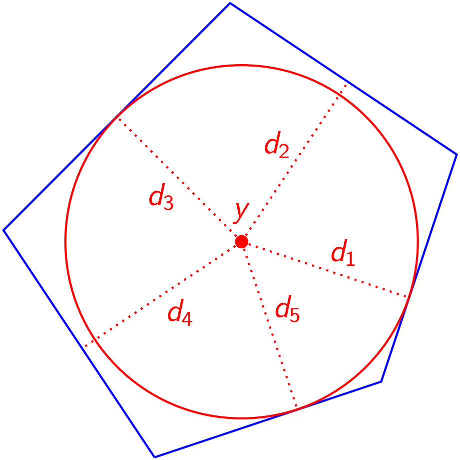
We want to maximize the smallest d_i. In other words,

$$\max_{y} \min_{i=1,\dots,5} d_i(y)$$

(the y shown here is obviously not optimal!)

Chebyshev center

What is the largest sphere you can fit inside a polyhedron?



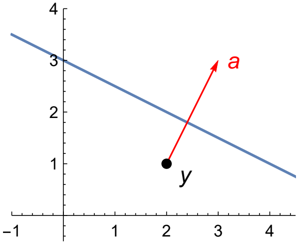
If y is the center, then draw perpendicular lines to each face of the polyhedron.

We want to maximize the smallest d_i. In other words,

$$\max_{y} \min_{i=1,\dots,5} d_i(y)$$

The optimal y is the Chebyshev center

Chebyshev center



To compute the distance between y and the hyperplane $a^\mathsf{T} x = b$, notice that if the distance is r, then

$$y + r\,\frac{a}{\|a\|}$$

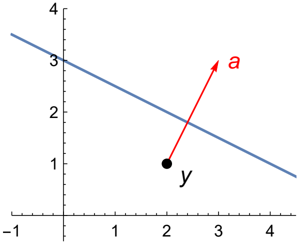
belongs to the hyperplane.

Recall that $\|a\| := \sqrt{a^\mathsf{T} a}$ (vector length) so $\frac{a}{\|a\|}$ is the unit vector in the direction of a.

Since this point belongs to the hyperplane, it satisfies $a^\mathsf{T} x = b$. Therefore, $a^\mathsf{T}\left(y + \frac{r}{\|a\|} a\right) = b$.

Simplifying, we obtain: $a^\mathsf{T} y + \|a\| r = b$

Chebyshev center



If the distance between y and the hyperplane $a^{\mathsf{T}}x = b$ is r, then $a^{\mathsf{T}}y + \|a\|r = b$.

If the distance is **at least** r, then $a^{\mathsf{T}}y + \|a\|r \le b$.

"The distance between y and each hyperplane is at least r" is equivalent to saying that $a_i^{\mathsf{T}}y + \|a_i\|r \le b_i$ for each i.

Finding the Chebyshev center amounts to solving an LP!

Chebyshev center

Finding the Chebyshev center amounts to solving an LP!

The transformation to an LP is given by:

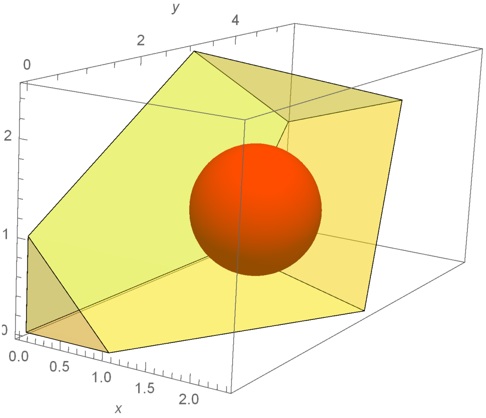
$$\begin{array}{ll} \max\limits_{y} & \min\limits_{i=1,\dots,k} d_i(y) \\ \text{s.t.} & a_i^T y \le b_i \quad \forall i \end{array} \quad \Longrightarrow \quad \begin{array}{ll} \max\limits_{y,r} & r \\ \text{s.t.} & a_i^T y + \|a_i\| r \le b_i \quad \forall i \end{array}$$

Chebyshev center

Example: find the Chebyshev center of the polyhedron defined by the following inequalities:

$$2x - y + 2z \leq 2, \ \ -x + 2y + 4z \leq 16, \ \ x + 2y - 2z \leq 8,$$
$$x \geq 0, \ \ y \geq 0, \ \ z \geq 0$$

Chebyshev.ipynb



Multi-period planning problems

- Optimization problems with a **temporal** component.
- Decisions must be made over the course of multiple time periods in order to optimize an overall cost.

Examples:

- scheduling: classes, tasks, employees, projects,...
- sequential decisions: investments, commitments,...

The decisions at each time period are **coupled** and must be jointly optimized. Otherwise we risk making decisions that seem good at the time but end up being very costly later.

Multi-period planning problems

- These problems tend to be tricky to model. It is often not clear what the decision variables should be.
- There are often more variables than you expect.

Important: Decision variables aren't always things that you decide directly!

We will see several examples of this...

Example: building a house

Several tasks must be completed in order to build a house.

- Each task takes a known amount of time to complete.
- A task may depend on other tasks, and can only be started once those tasks are complete.
- Tasks may be worked on simultaneously as long as they don't depend on one another.
- How fast can the house be built?

Job No.	Description	Immediate predecessors	Normal time (days)
a	Start		0
b	Excavate and pour footers	*a*	4
c	Pour concrete foundation	*b*	2
d	Erect wooden frame including rough roof	*c*	4
e	Lay brickwork	*d*	6
f	Install basement drains and plumbing	*c*	1
g	Pour basement floor	*f*	2
h	Install rough plumbing	*f*	3
i	Install rough wiring	*d*	2
j	Install heating and ventilating	*d,g*	4
k	Fasten plaster board and plaster (including drying)	*i,j,h*	10
l	Lay finish flooring	*k*	3
m	Install kitchen fixtures	*l*	1
n	Install finish plumbing	*l*	2
o	Finish carpentry	*l*	3
p	Finish roofing and flashing	*e*	2
q	Fasten gutters and downspouts	*p*	1
r	Lay storm drains for rain water	*c*	1
s	Sand and varnish flooring	*o,t*	2
t	Paint	*m,n*	3
u	Finish electrical work	*t*	1
v	Finish grading	*q,r*	2
w	Pour walks and complete landscaping	*v*	5
x	Finish	*s,u,w*	0

Source: HBR 1963

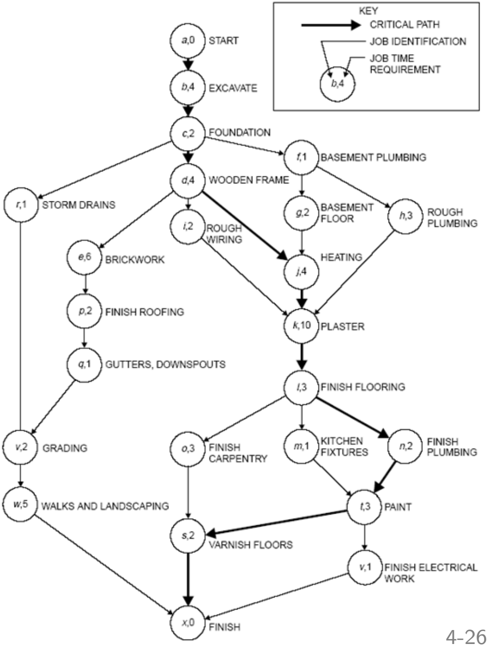
Example: building a house

The data can be visualized using a directed graph.

- Arrows indicate task dependencies.

> What are the decision variables?

- t_i: **start time** of i^{th} task.
- precedence constraints are expressed in terms of t_i's.
- minimize t_x.



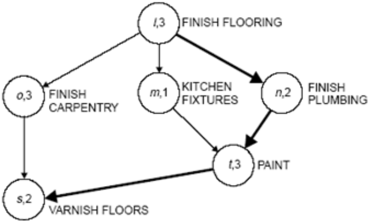
Source: HBR 1963

Example: building a house

A small sample:

Let t_l, t_o, t_m, t_n, t_t, t_s be start times of the associated tasks.

Now use the graph to write the dependency constraints:



- Tasks o, m, and n can't start until task l is finished, and task l takes 3 days to finish. So the constraints are:
$$t_l + 3 \le t_o, \quad t_l + 3 \le t_m, \quad t_l + 3 \le t_n$$
- Task t can't start until tasks m and n are finished. Therefore:
$$t_m + 1 \le t_t, \quad t_n + 2 \le t_t,$$
- Task s can't start until tasks o and t are finished. Therefore:
$$t_o + 3 \le t_s, \quad t_t + 3 \le t_s$$

Source: HBR 1963

Example: building a house

Full implementation in Julia:

House.ipynb

Critical path

Which tasks in the project can withstand delays?

- When $j \to i$, if we have $t_i = t_j + d_j$, then
  Task i must start immediately after task j is complete!
- If $t_i > t_j + d_j$, then we can withstand some delays in the completion of task j without delaying task i.

There must always exist a path in the graph where all constraints are at equality. This is called the **critical path**.

- If no such path existed, then we could complete the project more quickly (solution is not optimal)
- Related to notion of **duality** we will see later.