22. More models, QAP and SOS

- Quadratic assignment problems
- SOS1 constraints
- SOS2 constraints
- General piecewise linear functions

Quadratic assignment problems (QAP)

Two big classes of problems seen so far:

- **GAP:** Assign jobs to machines. Each assignment has a price, each machine has a fixed cost and a max capacity.
- **TSP:** Find the optimal sequence of items (or cities). Each possible transition has an associated cost. (probably the most famous integer program)

One more important class of problems:

- **QAP:** Assign facilities to fixed locations. The facilities have known communication requirements and incurred costs depend on the distances between the facilities. (second most famous integer program – harder than TSP)

QAP example: shopping mall

A small shopping mall has four shop locations. The walking distance, in feet, between all pairs of locations are shown below. Four shops, designated A, B, C, D, are to be assigned to the four locations in such a way that customers traveling between pairs of shops will not walk too far. We have data on the number of customers per week that travel between the shops, shown below.

Distance	1	2	3	4
1	0	80	150	170
2		0	130	100
3			0	120
4	d_{ij}			0

Flow	A	B	C	D
A	0	5	2	7
B		0	3	8
C			0	3
D	f_{ij}			0

- let $x_{ij} = 1$ if shop i is in location j.

QAP example: shopping mall

- Each shop can only be in one location:

$$\sum_{j=1}^{L} x_{ij} = 1 \qquad \text{for } i = 1, \ldots, S$$

- Each location can only contain one shop:

$$\sum_{i=1}^{S} x_{ij} = 1 \qquad \text{for } j = 1, \ldots, L$$

- Cost depends on pairs of facilities! If shop i is in location j and shop k is in location ℓ, then between them we have a flow of f_{ik} and a distance of $d_{j\ell}$. The total cost is:

$$\frac{1}{2} \sum_{i=1}^{S} \sum_{j=1}^{L} \sum_{k=1}^{S} \sum_{\ell=1}^{L} f_{ik} d_{j\ell} x_{ij} x_{k\ell}$$

QAP example: shopping mall

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & \frac{1}{2}\sum_{i=1}^{S}\sum_{j=1}^{L}\sum_{k=1}^{S}\sum_{\ell=1}^{L} f_{ik}\, d_{j\ell}\, x_{ij}\, x_{k\ell} \\
\text{subject to:} \quad & \sum_{j=1}^{L} x_{ij} = 1 \qquad \text{for } i = 1, \ldots, S \\
& \sum_{i=1}^{S} x_{ij} = 1 \qquad \text{for } j = 1, \ldots, L \\
& x_{ij} \in \{0, 1\}
\end{aligned}
$$

- cost is quadratic in the variables! Can we linearize?
- define new binary variable: $z_{ijk\ell} = x_{ij} x_{k\ell}$

QAP example: shopping mall

$$\begin{aligned}
\underset{x,z}{\text{minimize}} \quad & \frac{1}{2}\sum_{i=1}^{S}\sum_{j=1}^{L}\sum_{k=1}^{S}\sum_{\ell=1}^{L} f_{ik}\, d_{j\ell}\, z_{ijk\ell} \\
\text{subject to:} \quad & \sum_{j=1}^{L} x_{ij} = 1 \qquad \text{for } i = 1, \ldots, S \\
& \sum_{i=1}^{S} x_{ij} = 1 \qquad \text{for } j = 1, \ldots, L \\
& x_{ij} \in \{0, 1\} \\
& z_{ijk\ell} = x_{ij} x_{k\ell}
\end{aligned}$$

- Equivalent to: $(z_{ijk\ell} = 1) \iff (x_{ij} = 1) \wedge (x_{k\ell} = 1)$

QAP example: shopping mall

How do we model: $(z_{ijk\ell} = 1) \iff (x_{ij} = 1) \wedge (x_{k\ell} = 1)$?

- we saw this when we discussed logic constraints!
  - ($\Longrightarrow$): $x_{ij} \geq z_{ijk\ell}$ and $x_{k\ell} \geq z_{ijk\ell}$
  - ($\Longleftarrow$): $x_{ij} + x_{k\ell} \leq z_{ijk\ell} + 1$

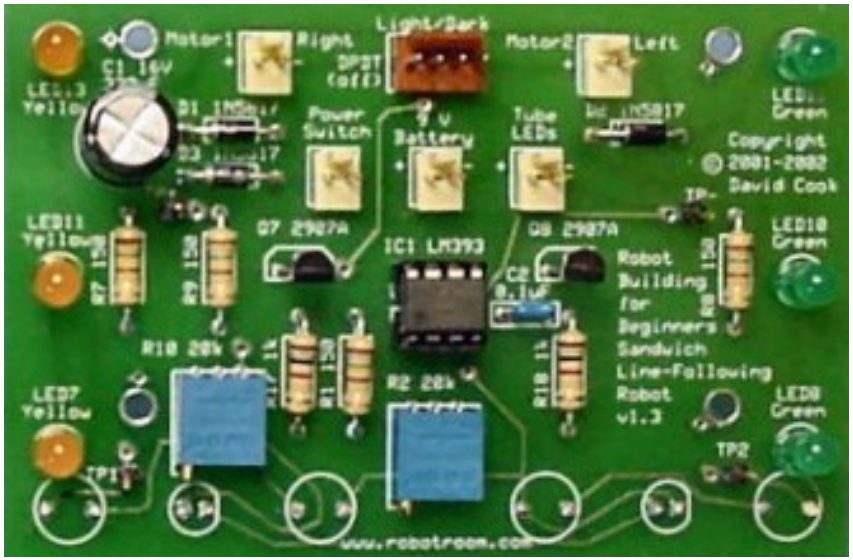
QAP example: shopping mall

$$
\begin{aligned}
\underset{x,z}{\text{minimize}} \quad & \frac{1}{2}\sum_{i=1}^{S}\sum_{j=1}^{L}\sum_{k=1}^{S}\sum_{\ell=1}^{L} f_{ik}\, d_{j\ell}\, z_{ijk\ell} \\
\text{subject to:} \quad & \sum_{j=1}^{L} x_{ij} = 1 && \forall i \\
& \sum_{i=1}^{S} x_{ij} = 1 && \forall j \\
& x_{ij} \geq z_{ijk\ell} \quad \text{and} \quad x_{k\ell} \geq z_{ijk\ell} && \forall i,j,k,\ell \\
& x_{ij} + x_{k\ell} \leq z_{ijk\ell} + 1 && \forall i,j,k,\ell \\
& x_{ij}, z_{ijk\ell} \in \{0,1\} && \forall i,j,k,\ell
\end{aligned}
$$

- Julia code: QAP.ipynb

QAP example: circuit layout

- Place n electronic modules in n predetermined positions on a backplate.
- We are given a wiring specification that tells us how the various modules must be connected.



- Identical to the facility location problem:
  - f_{ij} is the number of wires between module i and module j
  - d_{ij} distance between positions i and j on the backplate.
  - Minimize total length of wire used.

Special ordered sets

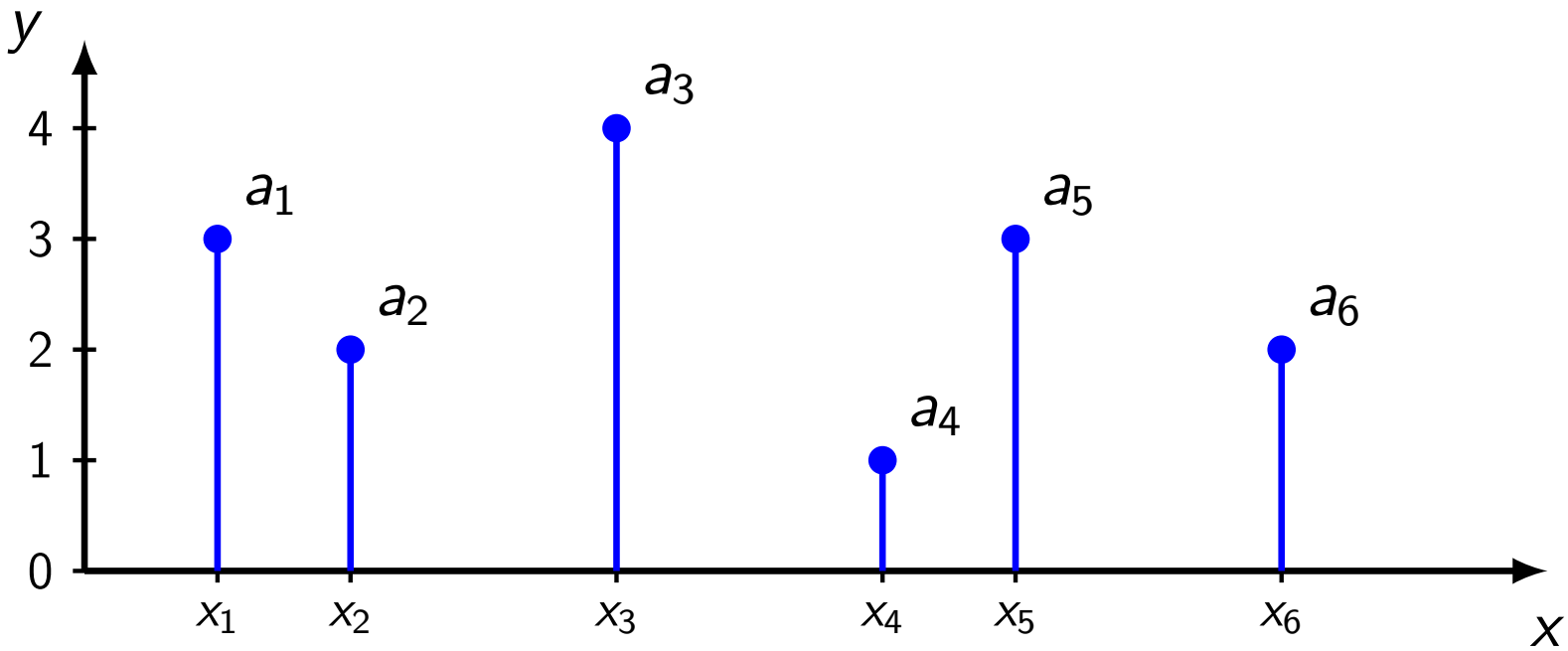
- Another type of constraint that is standard among many solvers is called the **special ordered set** (SOS).
- We saw one type of SOS constraint in the context of a variable belonging to a discrete set of values. There are other types of SOS constraints (we will see them next!)
- All SOS constraints can be implemented using logic tricks but some solvers also allow you to specify them explicitly. Only two reasons you would ever want to do this
  - It makes your code run faster
  - It makes your code easier to understand
- Some solvers can automatically detect SOS constraints.

SOS1 (type 1) constraint

SOS1 constraint: The variables $\{x_1, \ldots, x_m\}$ satisfy an SOS1 constraint if *at most one of them is nonzero*.

- An SOS1 constraint represents a multiple-choice notion.
- Standard use: representing a discrete-valued function

SOS1 (type 1) constraint



- Let (x_i, a_i), $i = 1, \ldots, m$ be the admissible (x, y) pairs.
- write: $x = \sum_{i=1}^{m} x_i \lambda_i$ and $y = \sum_{i=1}^{m} a_i \lambda_i$.
- constraint: exactly one of the λ_i is equal to 1, the rest are equal to zero.

SOS1 constraint

Using SOS1 constraint

$$y = \sum_{i=1}^{m} a_i \lambda_i$$

$$\sum_{i=1}^{m} \lambda_i = 1, \quad \lambda_i \geq 0$$

$\{\lambda_1, \ldots, \lambda_m\}$ is SOS1

Using algebraic forumlation

$$y = \sum_{i=1}^{m} a_i \lambda_i$$

$$\sum_{i=1}^{m} \lambda_i = 1$$

$$\lambda_i \in \{0, 1\}$$

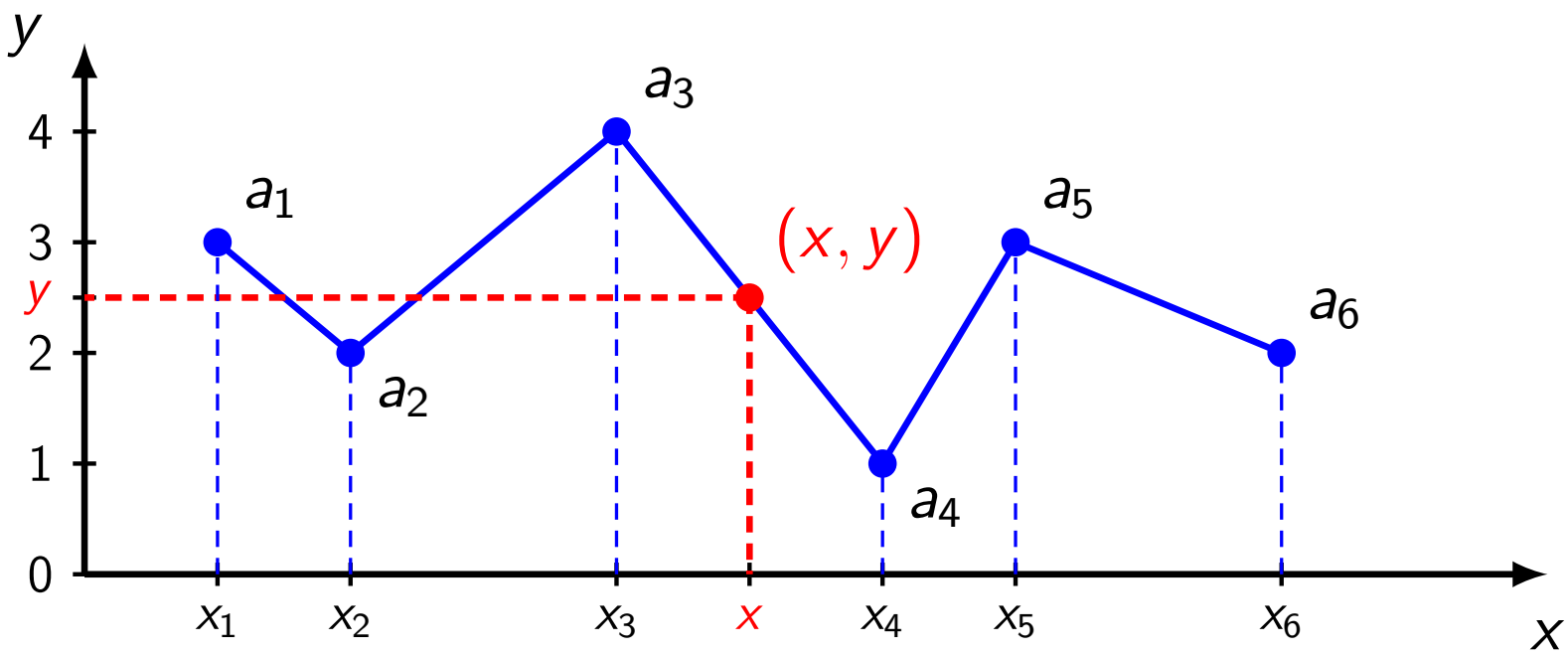
- The λ_i are real (not binary) in the SOS1 formulation
- Solver will still use binary variables internally, of course
- IJulia example: SOS.ipynb

SOS2 (type 2) constraint

SOS2 constraint: The variables $\{x_1, \ldots, x_m\}$ satisfy an SOS2 constraint if *at most two of them are nonzero*. Also, nonzero elements must be consecutive.

- SOS2 constraints are typically used to represent piecewise-linear functions.

SOS2 (type 2) constraint

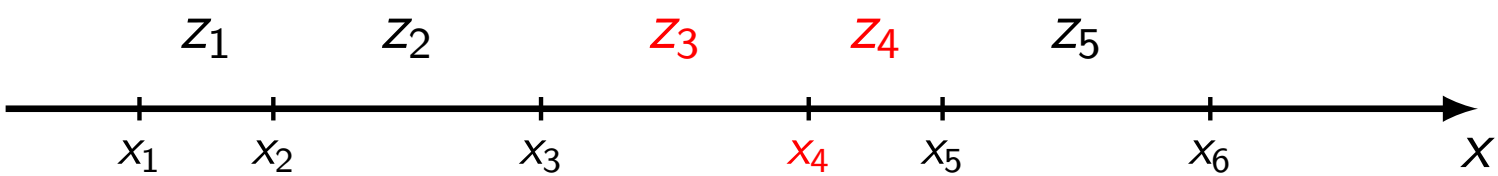


- Let (x_i, a_i), $i = 1, \ldots, m$ be the transition points.
- $x = \sum_{i=1}^{m} x_i \lambda_i$ and $y = \sum_{i=1}^{m} a_i \lambda_i$ with $\sum_{i=1}^{m} \lambda_i = 1$.
- If $x_i < x < x_{i+1}$ then $\lambda_i + \lambda_{i+1} = 1$ and the other λ_j are zero. Then, $x = \lambda_i x_i + \lambda_{i+1} x_{i+1}$ and $y = \lambda_i a_i + \lambda_{i+1} a_{i+1}$

SOS2 (type 2) constraint

How do we represent the constraint that at most two of the variables $\{\lambda_1, \ldots, \lambda_m\}$ are nonzero, and nonzero variables must be consecutive?

- Let $\{z_1, \ldots, z_{m-1}\}$ be binary with $\sum_{i=1}^{m-1} z_i = 1$.
- If $z_i = 1$, then λ_i and λ_{i+1} can be nonzero.
- In other words: if z_{i-1} and z_i are zero, then $\lambda_i = 0$.



SOS2 constraint

SOS2 constraint

$$y = \sum_{i=1}^{m} a_i \lambda_i$$

$$\sum_{i=1}^{m} \lambda_i = 1, \quad \lambda_i \geq 0$$

$\{\lambda_1, \ldots, \lambda_m\}$ is SOS2

- Extra binary variables needed in algebraic formulation.
- Solver still uses binary variables for SOS2.

Using algebraic forumlation

$$y = \sum_{i=1}^{m} a_i \lambda_i$$

$$\sum_{i=1}^{m} \lambda_i = 1, \qquad \lambda_i \geq 0$$

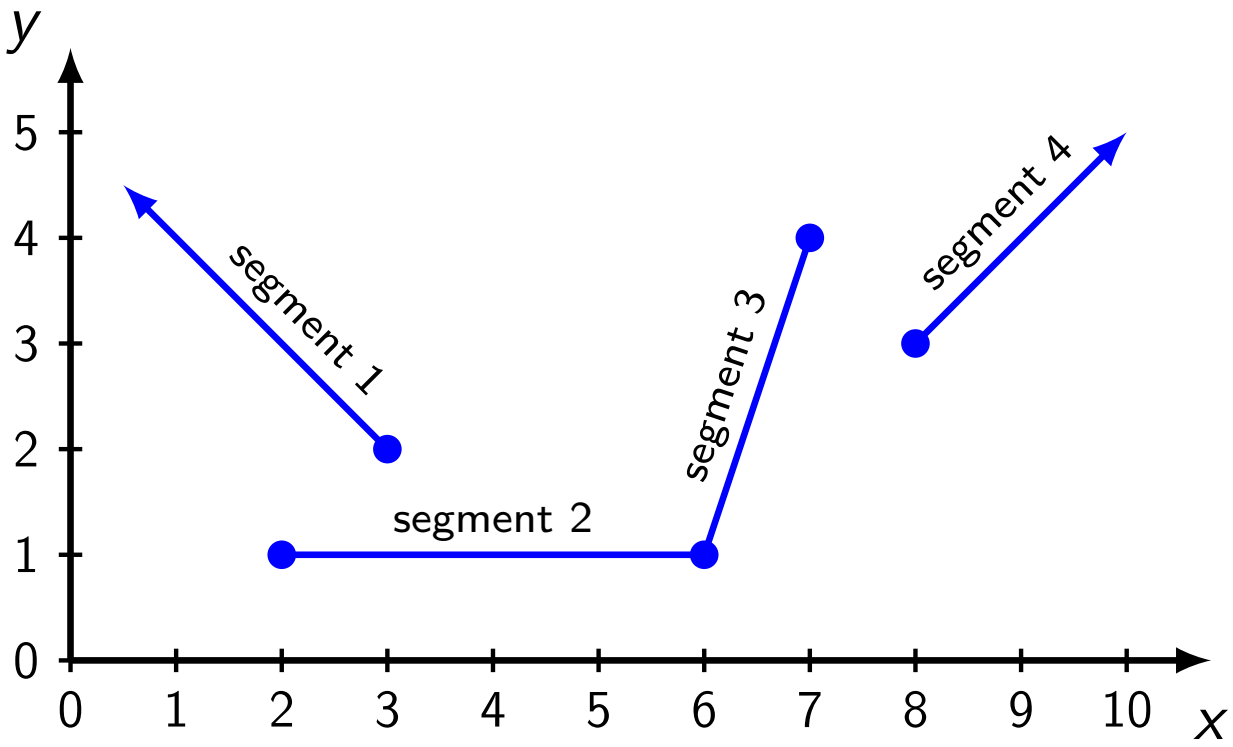
$$\lambda_1 \leq z_1$$

$$\lambda_i \leq z_{i-1} + z_i, \quad i = 2, \ldots, m-1$$

$$\lambda_m \leq z_{m-1}$$

$$\sum_{i=1}^{m-1} z_i = 1, \qquad z_i \in \{0, 1\}$$

General piecewise linear functions



We may have:

- segments overlapping
- semi-infinite or infinite segments
- holes in the space of x coordinates

For each segment, record:

- an endpoint (x, f)
- the x-length ℓ
- the slope g.

	x	f	ℓ	g
segment 1	3	2	$-\infty$	-1
segment 2	2	1	4	0
segment 3	6	1	1	3
segment 4	8	2	∞	1

General piecewise linear functions

- The segment i has two variables associated with it:
  - z_i, a binary variable that selects whether segment i is active or not. Note that the $\{z_1, \ldots, z_m\}$ are SOS1.
  - λ_i, a nonnegative real variable that chooses how far along segment i we are situated. Note that $0 \leq \lambda_i \leq |\ell_i|$.
- The general point (x, f) on the function is given by:
  - $x = \sum_{i=1}^{m}(z_i x_i + \text{sign}(\ell_i)\lambda_i)$, $f = \sum_{i=1}^{m}(z_i f_i + \text{sign}(\ell_i) g_i \lambda_i)$
- We also require some constraints:
  - $z_i \in \{0, 1\}$ and $\sum_{i=1}^{m} z_i = 1$
  - $0 \leq \lambda_i \leq |\ell_i|$.
  - if $z_i = 0$ then $\lambda_i = 0$. (this is tricky!)

General piecewise linear functions

How do we impose the constraint that if $z_i = 0$ then $\lambda_i = 0$?

- If ℓ_i is finite, then we have the bound $0 \leq \lambda_i \leq |\ell_i|$ so we can use the standard fixed cost trick:

$$\lambda_i \leq |\ell_i| z_i$$

- If $\ell_i = \pm\infty$ then this won't work!
- For the infinte case, we can use the constraint:

$$\{\lambda_i, 1 - z_i\} \text{ is SOS1}$$

There is no upper bound for λ_i (and so we can't model algebraically) but solvers can nonetheless implement this constraint efficiently.