15. Duality

- Upper and lower bounds
- General duality
- Constraint qualifications
- Counterexample
- Sensitivity analysis
- KKT conditions

Laurent Lessard (www.laurentlessard.com)

Upper bounds

Optimization problem (not necessarily convex!):

$$\begin{aligned} \underset{x \in D}{\text{minimize}} \quad & f_0(x) \\ \text{subject to:} \quad & f_i(x) \le 0 \quad \text{for } i = 1, \dots, m \\ & h_j(x) = 0 \quad \text{for } j = 1, \dots, r \end{aligned}$$

- D is the domain of all functions involved.
- Suppose the optimal value is $p^\star$.
- **Upper bounds:** if $x \in D$ satisfies $f_i(x) \le 0$ and $h_j(x) = 0$ for all i and j, then: $p^\star \le f_0(x)$.
- Any feasible x yields an upper bound for $p^\star$.

Lower bounds

Optimization problem (not necessarily convex!):

$$\begin{aligned} \underset{x \in D}{\text{minimize}} \quad & f_0(x) \\ \text{subject to:} \quad & f_i(x) \le 0 \quad \text{for } i = 1, \dots, m \\ & h_j(x) = 0 \quad \text{for } j = 1, \dots, r \end{aligned}$$

- As with LPs, use the constraints to find lower bounds
- For any $\lambda_i \ge 0$ and $\nu_j \in \mathbb{R}$, if $x \in D$ is feasible, then

$$f_0(x) \ge f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{r} \nu_j h_j(x)$$

Lower bounds

$$f_0(x) \geq \underbrace{f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{r} \nu_j h_j(x)}_{\text{Lagrangian } L(x,\lambda,\nu)}$$

This is a lower bound on f_0, but we want a lower bound on $p^\star$.

$$p^\star = \inf_{x \text{ feas}} f_0(x) \geq \inf_{x \text{ feas}} L(x,\lambda,\nu) \geq \underbrace{\inf_{x \in D} L(x,\lambda,\nu)}_{g(\lambda,\nu)}$$

This inequality holds whenever $\lambda \geq 0$.

Lower bounds

$$L(x, \lambda, \nu) := f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{r} \nu_j h_j(x)$$

Whenever $\lambda \geq 0$, we have:

$$g(\lambda, \nu) := \left\{ \inf_{x \in D} \; L(x, \lambda, \nu) \right\} \leq p^\star$$

Useful fact: $g(\lambda, \nu)$ is a **concave** function. This is true even if the original optimization problem is not convex! (because g is a pointwise minimum of affine functions)

General duality

Primal problem (P)

$$\begin{aligned} \underset{x \in D}{\text{minimize}} \quad & f_0(x) \\ \text{subject to:} \quad & f_i(x) \leq 0 \quad \forall i \\ & h_j(x) = 0 \quad \forall j \end{aligned}$$

Dual problem (D)

$$\begin{aligned} \underset{\lambda, \nu}{\text{maximize}} \quad & g(\lambda, \nu) \\ \text{subject to:} \quad & \lambda \geq 0 \end{aligned}$$

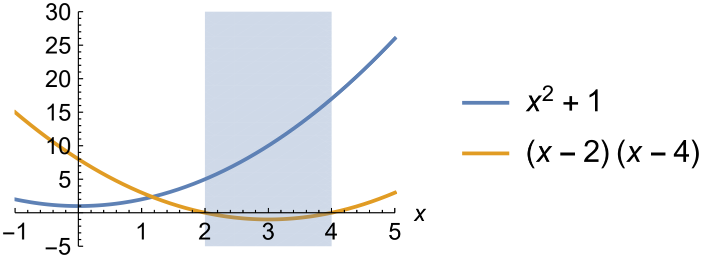
If x and λ are feasible points of (P) and (D) respectively:

$$g(\lambda, \nu) \leq d^\star \leq p^\star \leq f_0(x)$$

This is called the **Lagrange dual**. Bad news: strong duality ($p^\star = d^\star$) does **not** always hold!

Example (Srikant)

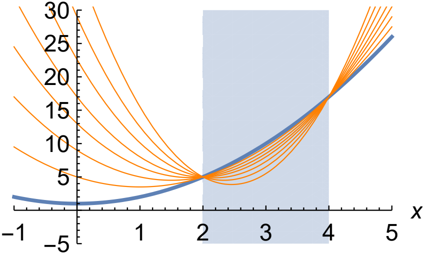
$$\begin{array}{ll} \underset{x}{\text{minimize}} & x^2 + 1 \\ \text{subject to:} & (x-2)(x-4) \leq 0 \end{array}$$



- optimum occurs at $x = 2$, has value $p^\star = 5$

Example (Srikant)

Lagrangian: $L(x, \lambda) = x^2 + 1 + \lambda(x-2)(x-4)$



- Plot for different values of $\lambda \geq 0$
- $g(\lambda) = \inf_x L(x, \lambda)$ should be a lower bound on $p^\star = 5$ for all $\lambda \geq 0$.

Example (Srikant)

Lagrangian: $L(x,\lambda) = x^2 + 1 + \lambda(x-2)(x-4)$

- Minimize the Lagrangian:

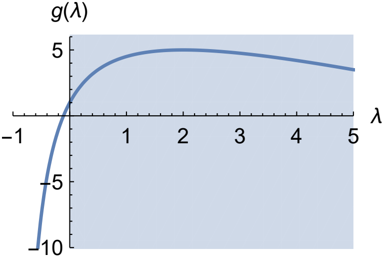
$$\begin{aligned} g(\lambda) &= \inf_x \; L(x,\lambda) \\ &= \inf_x \; (\lambda+1)x^2 - 6\lambda x + (8\lambda+1) \end{aligned}$$

If $\lambda \le -1$, it is unbounded. If $\lambda > -1$, the minimum occurs when $2(\lambda+1)x - 6\lambda = 0$, so $\hat{x} = \frac{3\lambda}{\lambda+1}$.

$$g(\lambda) = \begin{cases} -9\lambda^2/(1+\lambda) + 1 + 8\lambda & \lambda > -1 \\ -\infty & \lambda \le -1 \end{cases}$$

Example (Srikant)

$$\begin{aligned} \underset{\lambda}{\text{maximize}} \quad & -9\lambda^2/(1+\lambda) + 1 + 8\lambda \\ \text{subject to:} \quad & \lambda \geq 0 \end{aligned}$$



- optimum occurs at $\lambda = 2$, has value $d^\star = 5$
- same optimal value as primal problem! (strong duality)

Constraint qualifications

- weak duality ($d^\star \leq p^\star$) always holds. Even when the optimization problem is not convex.
- strong duality ($d^\star = p^\star$) often holds for convex problems (but not always).

A **constraint qualification** is a condition that guarantees strong duality. An example we've already seen:

- If the optimization problem is an LP, strong duality holds

Slater's constraint qualification

$$\begin{aligned} \underset{x \in D}{\text{minimize}} \quad & f_0(x) \\ \text{subject to:} \quad & f_i(x) \leq 0 \quad \text{for } i = 1, \dots, m \\ & h_j(x) = 0 \quad \text{for } j = 1, \dots, r \end{aligned}$$

Slater's constraint qualification:

> If the optimization problem is convex and strictly feasible, then strong duality holds.

- convexity requires: D and f_i are convex and h_j are affine.
- strict feasibility means there exists some $\tilde{x}$ in the interior of D such that $f_i(\tilde{x}) < 0$ for $i = 1, \dots, m$.

Slater's constraint qualification

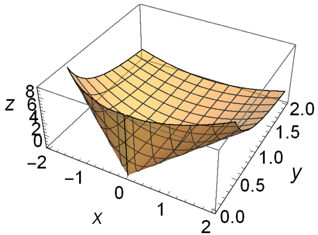
If the optimization problem is convex and strictly feasible, then strong duality holds.

- Good news: Slater's constraint qualification is rather weak. i.e. it is usually satisfied by convex problems.
- Can be relaxed so that strict feasibility is not required for any of the f_i that happen to be affine.

Counterexample (Boyd)

$$\begin{aligned} \underset{x\in\mathbb{R},\, y>0}{\text{minimize}} \quad & e^{-x} \\ \text{subject to:} \quad & x^2/y \le 0 \end{aligned}$$

- The function x^2/y is convex for $y > 0$ (see plot)
- The objective e^{-x} is convex
- Feasible set: $\{(0, y) \mid y > 0\}$
- Solution is trivial ($p^\star = 1$)



Counterexample (Boyd)

$$\begin{array}{ll} \underset{x\in\mathbb{R},\, y>0}{\text{minimize}} & e^{-x} \\ \text{subject to:} & x^2/y \leq 0 \end{array}$$

- **Lagrangian:** $L(x, y, \lambda) = e^{-x} + \lambda x^2/y$
- **Dual function:** $g(\lambda) = \inf_{x,y>0} \left(e^{-x} + \lambda x^2/y\right) = 0.$
- The dual problem is:

$$\underset{\lambda \geq 0}{\text{maximize}} \quad 0$$

So we have $d^\star = 0 < 1 = p^\star$.

- Slater's constraint qualification is **not** satisfied!

About Slater's constraint qualification

Slater's condition is only **sufficient**.

(Slater) $\implies$ (strong duality)

- There exist problems where Slater's condition fails, yet strong duality holds.
- There exist nonconvex problems with strong duality.

Dual of an LP

$$\begin{aligned} \underset{x \geq 0}{\text{minimize}} \quad & c^{\mathsf{T}} x \\ \text{subject to:} \quad & Ax \geq b \end{aligned}$$

- Lagrangian: $L(x, \lambda) = c^{\mathsf{T}} x + \lambda^{\mathsf{T}}(b - Ax)$
- Dual function: $g(\lambda) = \min_{x \geq 0} \left(c - A^{\mathsf{T}}\lambda\right)^{\mathsf{T}} x + \lambda^{\mathsf{T}} b$

$$g(\lambda) = \begin{cases} \lambda^{\mathsf{T}} b & \text{if } A^{\mathsf{T}}\lambda \leq c \\ -\infty & \text{otherwise} \end{cases}$$

Dual of an LP

$$\begin{aligned} \underset{x \geq 0}{\text{minimize}} \quad & c^{\mathsf{T}} x \\ \text{subject to:} \quad & Ax \geq b \end{aligned}$$

- Dual is:

$$\begin{aligned} \underset{\lambda \geq 0}{\text{maximize}} \quad & \lambda^{\mathsf{T}} b \\ \text{subject to:} \quad & A^{\mathsf{T}} \lambda \leq c \end{aligned}$$

- This is the same result that we found when we were studying duality for linear programs.

Dual of an LP

What if we treat $x \geq 0$ as a constraint instead? ($D = \mathbb{R}^n$).

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & c^\mathsf{T} x \\ \text{subject to:} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

- Lagrangian: $L(x, \lambda, \mu) = c^\mathsf{T} x + \lambda^\mathsf{T}(b - Ax) - \mu^\mathsf{T} x$
- Dual function: $g(\lambda, \mu) = \min_x (c - A^\mathsf{T}\lambda - \mu)^\mathsf{T} x + \lambda^\mathsf{T} b$

$$g(\lambda) = \begin{cases} \lambda^\mathsf{T} b & \text{if } A^\mathsf{T}\lambda + \mu = c \\ -\infty & \text{otherwise} \end{cases}$$

Dual of an LP

What if we treat $x \geq 0$ as a constraint instead? ($D = \mathbb{R}^n$).

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & c^\mathsf{T} x \\ \text{subject to:} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

- Dual is:

$$\begin{aligned} \underset{\lambda \geq 0,\, \mu \geq 0}{\text{maximize}} \quad & \lambda^\mathsf{T} b \\ \text{subject to:} \quad & A^\mathsf{T} \lambda + \mu = c \end{aligned}$$

- Solution is the same, μ acts as the slack variable.

Dual of a convex QP

Suppose $Q \succ 0$. Let's find the dual of the QP:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \tfrac{1}{2}x^\mathsf{T} Q x \\ \text{subject to:} \quad & Ax \geq b \end{aligned}$$

- Lagrangian: $L(x,\lambda) = \frac{1}{2}x^\mathsf{T} Q x + \lambda^\mathsf{T}(b - Ax)$
- Dual function: $g(\lambda) = \min_x \left(\frac{1}{2}x^\mathsf{T} Q x + \lambda^\mathsf{T}(b - Ax) \right)$
  Minimum occurs at: $\hat{x} = Q^{-1}A^\mathsf{T}\lambda$

$$g(\lambda) = -\tfrac{1}{2}\lambda^\mathsf{T} A Q^{-1} A^\mathsf{T} \lambda + \lambda^\mathsf{T} b$$

Dual of a convex QP

Suppose $Q \succ 0$. Let's find the dual of the QP:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \tfrac{1}{2}x^\mathsf{T} Q x \\ \text{subject to:} \quad & Ax \geq b \end{aligned}$$

- Dual is also a QP:

$$\begin{aligned} \underset{\lambda}{\text{maximize}} \quad & -\tfrac{1}{2}\lambda^\mathsf{T} A Q^{-1} A^\mathsf{T} \lambda + \lambda^\mathsf{T} b \\ \text{subject to:} \quad & \lambda \geq 0 \end{aligned}$$

- It's still easy to solve (maximizing a concave function)

Sensitivity analysis

$$\begin{array}{lll} \min\limits_{x \in D} & f_0(x) & \\ \text{s.t.} & f_i(x) \leq u_i & \forall i \\ & h_j(x) = v_j & \forall j \end{array}$$

$$\begin{array}{ll} \max\limits_{\lambda, \nu} & g(\lambda, \nu) - \lambda^\mathsf{T} u - \nu^\mathsf{T} v \\ \text{s.t.} & \lambda \geq 0 \end{array}$$

- As with LPs, dual variables quantify the sensitivity of the optimal cost to changes in each of the constraints.
- A change in u_i causes a bigger change in $p^\star$ if $\lambda_i^\star$ is larger.
- A change in v_j causes a bigger change in $p^\star$ if $\nu_j^\star$ is larger.
- If $p^\star(u, v)$ is differentiable, then:

$$\lambda_i^\star = -\frac{\partial p^\star(0,0)}{\partial u_i} \qquad \text{and} \qquad \nu_j^\star = -\frac{\partial p^\star(0,0)}{\partial v_j}$$

Karush-Kuhn-Tucker conditions

$$\begin{aligned} \underset{x \in D}{\text{minimize}} \quad & f_0(x) \\ \text{subject to:} \quad & f_i(x) \leq 0 \quad \text{for } i = 1, \ldots, m \\ & h_j(x) = 0 \quad \text{for } j = 1, \ldots, r \end{aligned}$$

Define the Lagrangian

$$L(x, \lambda, \nu) := f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{r} \nu_j h_j(x)$$

We say $(x^\star, \lambda^\star, \nu^\star)$ satisfies the KKT conditions if:

1. Stationarity: $\nabla_x L(x^\star, \lambda^\star, \nu^\star) = 0$.
2. Primal feasibility: $x^\star \in D$ and $f_i(x^\star) \leq 0$ and $h_j(x^\star) = 0$.
3. Dual feasibility: $\lambda_i^\star \geq 0$.
4. Complementary slackness: $\lambda_i^\star f_i(x^\star) = 0$.

Necessary condition for optimality

First version:

Theorem 1: Suppose $x^\star$ is primal optimal and $(\lambda^\star, \nu^\star)$ is dual optimal and there is no duality gap (strong duality). Then, $(x^\star, \lambda^\star, \nu^\star)$ satisfies the KKT conditions.

- Does not require that the problem is convex!

Second version:

Theorem 2: Suppose $x^\star$ is primal optimal, primal is convex, and Slater's condition holds. Then, there exists $(\lambda^\star, \nu^\star)$ such that $(x^\star, \lambda^\star, \nu^\star)$ satisfies the KKT conditions. Furthermore, any such $(\lambda^\star, \nu^\star)$ is dual optimal.

Sufficient condition for optimality

Theorem 3: Suppose the primal problem is convex and $(x^\star, \lambda^\star, \nu^\star)$ satisfies the KKT conditions. Then,

- $x^\star$ is primal optimal.
- $(\lambda^\star, \nu^\star)$ is dual optimal.
- strong duality holds.

Proof of necessity

Theorem 1: Suppose $x^\star$ is primal optimal and $(\lambda^\star, \nu^\star)$ is dual optimal and there is no duality gap (strong duality). Then, $(x^\star, \lambda^\star, \nu^\star)$ satisfies the KKT conditions.

We have primal and dual feasibility by assumption. It remains to prove stationarity and complementary slackness.

$$
\begin{aligned}
f_0(x^\star) = g(\lambda^\star, \nu^\star) &= \inf_{x \in D} \left(f_0(x) + \sum_i \lambda_i^\star f_i(x) + \sum_j \nu_j^\star h_j(x) \right) \\
&\leq f_0(x^\star) + \sum_i \lambda_i^\star f_i(x^\star) + \sum_j \nu_j^\star h_j(x^\star) \leq f_0(x^\star)
\end{aligned}
$$

Therefore, all are equal and we conclude $\lambda_i^\star f_i(x^\star) = 0$ for all i (complementary slackness), and $x^\star$ is a minimizer of $L(x, \lambda^\star, \nu^\star)$, so $\nabla_x L(x, \lambda^\star, \nu^\star) = 0$ (stationarity).

Proof of sufficiency

Theorem 3: Suppose the primal problem is convex and $(x^\star, \lambda^\star, \nu^\star)$ satisfies the KKT conditions. Then,

- $x^\star$ is primal optimal.
- $(\lambda^\star, \nu^\star)$ is dual optimal.
- strong duality holds.

The f_i are convex, the h_j are affine, and $\lambda^\star \geq 0$ (dual feasibility). So, $L(x, \lambda^\star, \nu^\star)$ is convex. By stationarity and comp. slackness,

$$g(\lambda^\star, \nu^\star) = f_0(x^\star) + \sum_i \lambda_i^\star f_i(x^\star) + \sum_j \nu_j^\star h_j(x^\star) = f_0(x^\star).$$

Therefore, we have zero duality gap. By primal feasibility and weak duality, $(x^\star, \lambda^\star, \nu^\star)$ is primal and dual optimal.

Summary: optimality

We say $(x^\star, \lambda^\star, \nu^\star)$ satisfies the KKT conditions if:

1. Stationarity: $\nabla_x L(x^\star, \lambda^\star, \nu^\star) = 0$.
2. Primal feasibility: $x^\star \in D$ and $f_i(x^\star) \leq 0$ and $h_j(x^\star) = 0$.
3. Dual feasibility: $\lambda_i^\star \geq 0$.
4. Complementary slackness: $\lambda_i^\star f_i(x^\star) = 0$.

For a convex optimization problem that satisfies Slater:

$$(x^\star, \lambda^\star, \nu^\star) \text{ is a primal/dual optimal solution}$$
$$\iff (x^\star, \lambda^\star, \nu^\star) \text{ satisfies the KKT conditions}$$

Example

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & x^2 + 1 \\ \text{subject to:} \quad & (x-2)(x-4) \leq 0 \end{aligned}$$

Lagrangian: $L(x, \lambda) = x^2 + 1 + \lambda(x-2)(x-4)$

KKT conditions:

$$\begin{aligned} 2x + \lambda(2x-6) &= 0 \quad \text{(stationarity)} \\ (x-2)(x-4) &\leq 0 \quad \text{(primal feasibility)} \\ \lambda &\geq 0 \quad \text{(dual feasibility)} \\ \lambda(x-2)(x-4) &= 0 \quad \text{(complementary slackness)} \end{aligned}$$

Solution: $x = 2$, $\lambda = 2$.

Example, part 2

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & x^2 + 1 \\ \text{subject to:} \quad & 2 \le x \le 4 \end{aligned}$$

Lagrangian: $L(x, \lambda, \mu) = x^2 + 1 + \lambda(2 - x) + \mu(x - 4)$

KKT conditions:

$$\begin{aligned} 2x - \lambda + \mu = 0 \quad & \text{(stationarity)} \\ 2 \le x \le 4 \quad & \text{(primal feasibility)} \\ \lambda \ge 0, \quad \mu \ge 0 \quad & \text{(dual feasibility)} \\ \lambda(2 - x) = 0, \quad \mu(x - 4) = 0 \quad & \text{(complementary slackness)} \end{aligned}$$

Solution: $x = 2$, $\lambda = 4$, $\mu = 0$.