10. Regularization

- More on tradeoffs
- Regularization
- Effect of using different norms
- Example: hovercraft revisited

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Review of tradeoffs

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- Each $\lambda > 0$ yields a solution $\hat{x}_\lambda$.
- Can visualize tradeoff by plotting $J_2(\hat{x}_\lambda)$ vs $J_1(\hat{x}_\lambda)$. This is called the **Pareto curve**.

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- Can visualize tradeoff by plotting $J_3(\hat{x}_{\lambda,\mu})$ vs $J_2(\hat{x}_{\lambda,\mu})$ vs $J_1(\hat{x}_{\lambda,\mu})$ on a 3D plot. You then obtain a **Pareto surface**.

Minimum-norm as a regularization

- When $Ax = b$ is underdetermined (A is wide), we can resolve ambiguity by adding a cost function, e.g. **min-norm** LS:

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Solution is found via pseudoinverse (for tall matrix)

$$\hat{x} = \left(\begin{bmatrix} A \\ \sqrt{\lambda} I \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} A \\ \sqrt{\lambda} I \end{bmatrix} \right)^{-1} \begin{bmatrix} A \\ \sqrt{\lambda} I \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} b \\ 0 \end{bmatrix}$$

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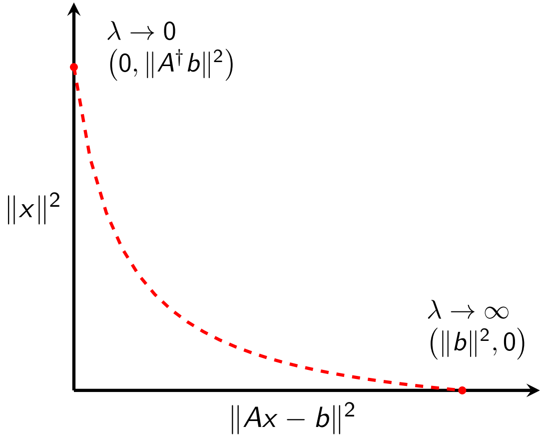
- Since AA^T is invertible, we can take the limit $\lambda \to 0$ by just setting $\lambda = 0$.
- In the limit: $\hat{x} = A^\mathsf{T}(AA^\mathsf{T})^{-1} b$. This is the exact solution to the minimum-norm least squares problem we found before!

Tradeoff visualization

$$\underset{x}{\text{minimize}} \quad \|Ax - b\|^2 + \lambda \|x\|^2$$



Regularization

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Regularized least squares:

$$\underset{x}{\text{minimize}} \quad \|Ax - b\|^2 + \lambda R(x)$$

- $R(x)$ is the regularizer (penalty function)
- λ is the regularization parameter
- The model has different names depending on $R(x)$.

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It is called: L_2 *regularization*, *Tikhonov regularization*, or *Ridge regression* depending on the application. It has the effect of **smoothing** the solution.

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3. $R(x) = \|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$
   It is called L_∞ *regularization* and it has the effect of **equalizing** the solution (makes most components equal).

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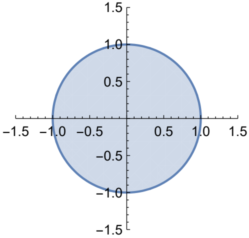

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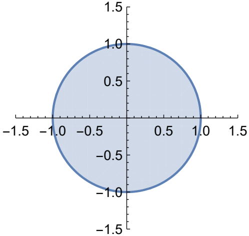

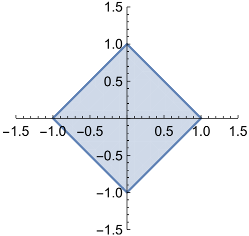

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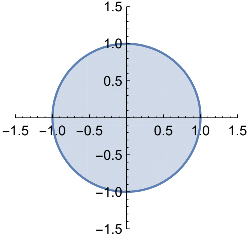

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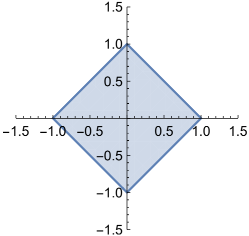

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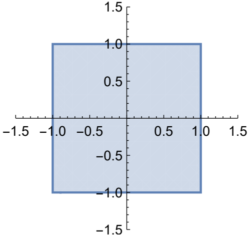

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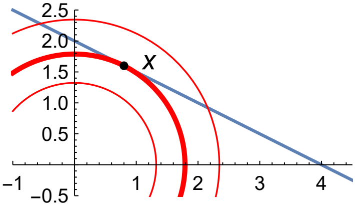
Simple example

Consider the minimum-norm problem for different norms:

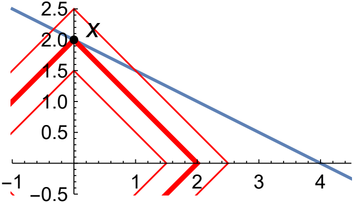
$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \|x\|_p \\ \text{subject to:} \quad & Ax = b \end{aligned}$$

- set of solutions to $Ax = b$ is an affine subspace
- solution is point belonging to smallest norm ball
- for $p = 2$, this occurs at the perpendicular distance



Simple example

- for $p = 1$, this occurs at one of the axes.
- sparsifying behavior



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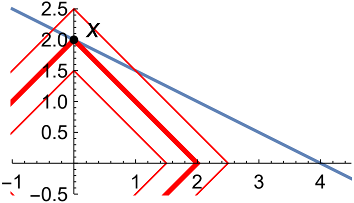
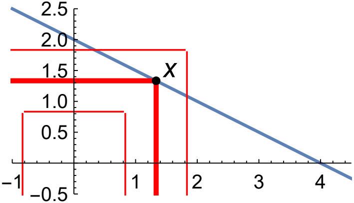


- for $p = \infty$, this occurs at equal values of coordinates
- equalizing behavior



Another simple example

Suppose we have data points $\{y_1, \dots, y_m\} \subset \mathbb{R}$, and we would like to find the best estimator for the data, according to different norms. Suppose data is sorted: $y_1 \leq \cdots \leq y_m$.

$$\underset{x}{\text{minimize}} \quad \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} - \begin{bmatrix} x \\ \vdots \\ x \end{bmatrix} \right\|_p$$

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Julia demo: Data Norm.ipynb

Example: hovercraft revisited

One-dimensional version of the hovercraft problem:

- Start at $x_1 = 0$ with $v_1 = 0$ (at rest at position zero)
- Finish at $x_{50} = 100$ with $v_{50} = 0$ (at rest at position 100)
- Same simple dynamics as before:

$$\begin{aligned} x_{t+1} &= x_t + v_t \\ v_{t+1} &= v_t + u_t \end{aligned} \qquad \text{for: } t = 1, 2, \dots, 49$$

- Decide thruster inputs $u_1, u_2, \dots, u_{49}$.
- This time: minimize $\|u\|_p$

Example: hovercraft revisited

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- This model has 150 variables, but very easy to understand.

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- This model has 150 variables, but very easy to understand.
- We can simplify the model considerably...

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$$\begin{aligned} v_{50} &= v_{49} + u_{49} \\ &= v_{48} + u_{48} + u_{49} \\ &= \dots \\ &= v_1 + \left(u_1 + u_2 + \cdots + u_{49}\right) \end{aligned}$$

Model simplification

$$x_{t+1} = x_t + v_t \qquad \text{for: } t = 1, 2, \dots, 49$$
$$v_{t+1} = v_t + u_t$$

$$\begin{aligned} x_{50} &= x_{49} + v_{49} \\ &= x_{48} + 2v_{48} + u_{48} \\ &= x_{47} + 3v_{47} + 2u_{47} + u_{48} \\ &= \dots \\ &= x_1 + 49v_1 + (48u_1 + 47u_2 + \cdots + 2u_{47} + u_{48}) \end{aligned}$$

Model simplification

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$$v_{t+1} = v_t + u_t$$

Constraint can be rewritten as:

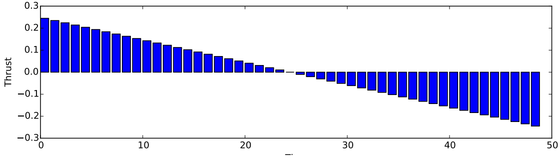
$$\begin{bmatrix} 48 & 47 & \dots & 2 & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{49} \end{bmatrix} = \begin{bmatrix} x_{50} - x_1 - 49v_1 \\ v_{50} - v_1 \end{bmatrix}$$

so we don't need the intermediate variables x_t and v_t!

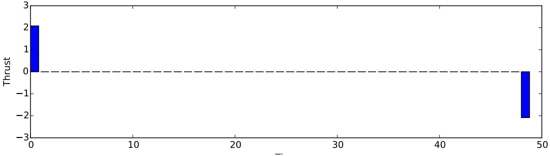
Julia demo: Hover 1D.ipynb

Results

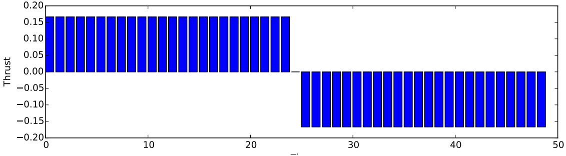
1. Minimizing $\|u\|_2^2$ (smooth)



2. Minimizing $\|u\|_1$ (sparse)

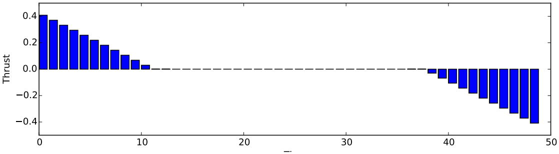


3. Minimizing $\|u\|_\infty$ (equalized)

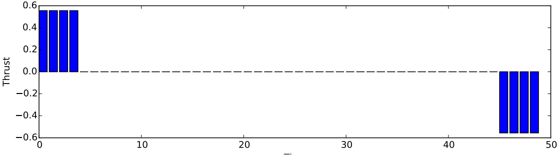


Tradeoff studies

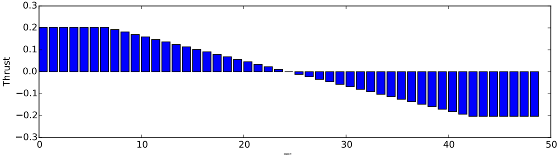
1. Minimizing $\|u\|_2^2 + \lambda\|u\|_1$ (smooth and sparse)



2. Minimizing $\|u\|_\infty + \lambda\|u\|_1$ (equalized and sparse)



3. Minimizing $\|u\|_2^2 + \lambda\|u\|_\infty$ (equalized and smooth)



Next class...

- Symmetric matrices and eigenvalues
- Quadratic forms and ellipsoids
- Convex vs nonconvex QPs
- Examples