Root Locus Plots

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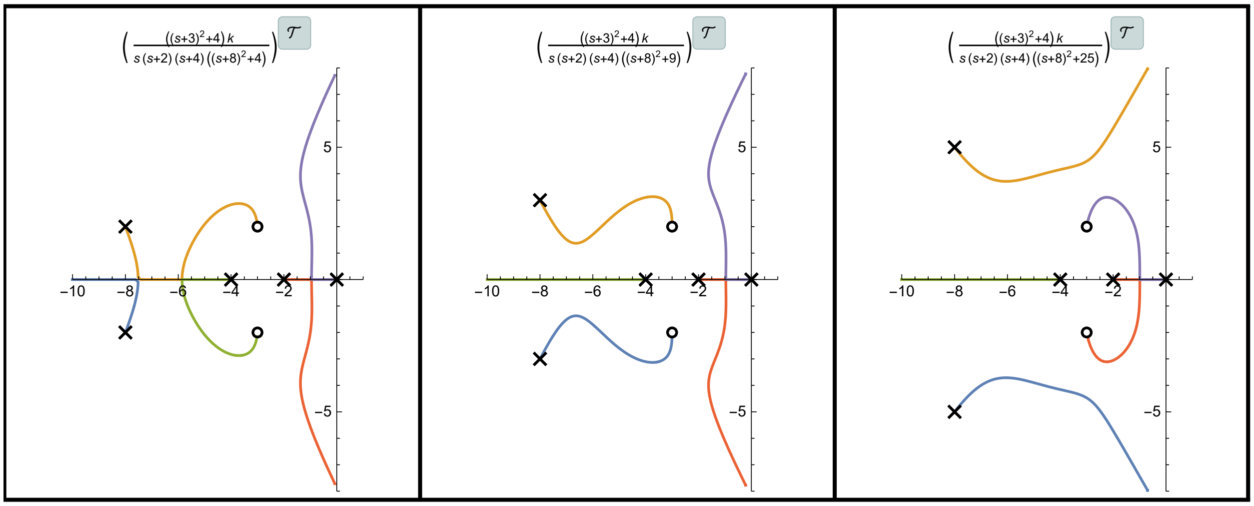
Here is a summary of the rules we saw for sketching root locus plots. If $G(s)$ has n poles $\{p_1, \dots, p_n\}$ and m zeros $\{z_1, \dots, z_m\}$ with $n \geq m$, then:

1. The root locus has n branches. A branch cannot cross itself.
2. The root locus is symmetric about the real axis.
3. Parts of the real axis left of an *odd* number of real poles or zeros belong to the root locus.
4. Each branch starts at a different open-loop pole. m of the branches end at different open-loop zeros. The remaining $n - m$ branches go to infinity.
5. The $n - m$ asymptotes are at angles $\frac{(2k+1)\pi}{m-n}$ for $k = 0, 1, 2, \dots$. The asymptotes intersect on the real axis at the centroid, located at $\sigma_A = \frac{1}{n-m}\left(\sum_i p_i - \sum_i z_i\right)$. Some intuition:
   - Moving a pole left or right, the centroid moves in the *same direction.* Likewise, adding a new pole left or right of the current centroid moves the centroid left or right, respectively.
   - Moving a zero left or right, the centroid moves in the *opposite direction.* Also, adding a new zero left or right of the current centroid moves the centroid right or left, respectively.
6. When two branches meet on the real axis, they will break at right angles to each other. This can happen in two ways:
   - The two branches start on the real axis moving toward each other. They collide and *break-away* perpendicular to the real axis, moving away from each other.
   - The two branches are complex conjugates, move toward each other, meet on the real axis, intersecting it at right angles. They *break-in* and continue along the real axis in opposite directions.

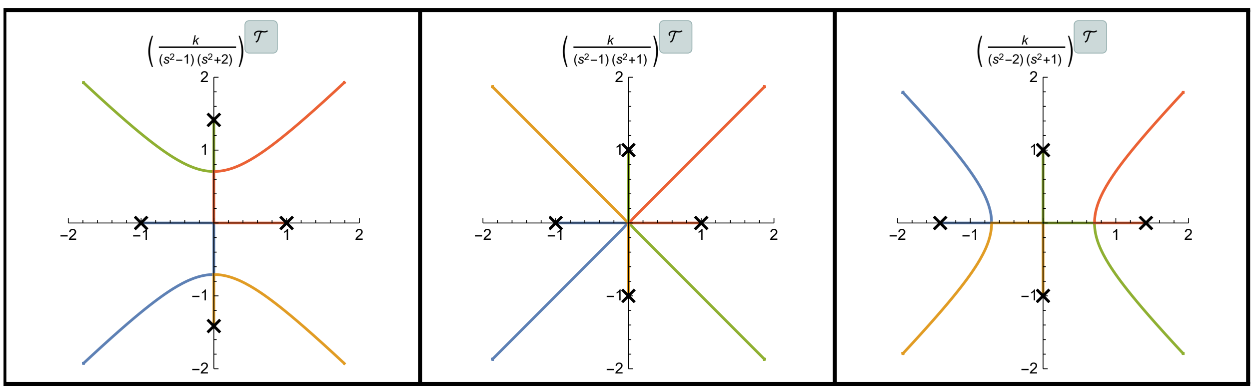
Some additional items we did *not* discuss in class:

- Calculating departure and arrival angles of branches at poles and zeros, respectively.
- Calculating the exact locations of break-away and break-in points on the real axis.
- Calculating where a root locus crosses the imaginary axis into instability.
- What happens when there are multiple poles or zeros in the same location.
- What happens when $K < 0$. This is called the *negative locus.*

It is not always easy to predict which poles will meet which zeros. The figure below illustrates three similar sets of poles and zeros. They have the same real locus, the same centroid, and the same asymptotes. And yet, their root loci look quite different.



We can have break-away points that are not on the real axis. You can also have a single break-away point where more than two poles meet. These are typically pathological scenarios; if we nudge any of the poles in any direction, we obtain a more typical-looking locus. Examples below.



We can also have break-in points that are not on the real axis. These are also pathological scenarios.