ME 4555 Refresher

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These problems cover some of the math concepts needed for ME 4555. They are for your personal reference; they will not be graded and are not to be turned in. The goal is to identify any concepts that might require additional review before the course gets going. Solutions are on the next page.

Calculus

1. Use the chain rule to compute: $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(e^{3t}\sin(5t)\right)$

2. Evaluate the integral: $\displaystyle\int_0^t \sin(6\tau)\,\mathrm{d}\tau$

3. Solve the differential equation: $y'(t) + 3y(t) = 0$, with the initial condition $y(0) = 2$.

Complex numbers

In the problems below, j is the imaginary unit; i.e. $j^2 = -1$.

1. Compute the real part $\mathrm{Re}(z)$ and the imaginary part $\mathrm{Im}(z)$ of the complex number $z = -2 + 3j$.

2. Compute the magnitude $|z|$ and argument $\arg(z)$ of $z = -1 + j$. For the argument, express your answer in both radians and degrees.

3. Draw the complex numbers $z_1 = 2 + 4j$ and $z_2 = -1 + j$ in the complex plane.

4. Compute the real and imaginary parts of $z = 3e^{j\pi}$.

5. Compute $|e^{j\omega}|$, where ω is a real number.

6. Compute $\arg(e^{j\omega})$, where ω is a real number. Express your answer in both radians and degrees.

7. Compute $\left|-2(-1+2j)(-4-3j)\right|$.

8. Write $\dfrac{-3+2j}{3-4j}$ in the form $a + bj$ for some real numbers a and b.

Polynomials

1. Find all solutions to $x^2 - x + 4 = 0$

2. Find all solutions to $3x^2 + 2x + 1 = 0$.

3. Simplify the expression $\dfrac{x^2 - 2x - 3}{x^2 + 3x + 2}$ by canceling common factors.

4. For what values of a (real number) does the equation $x^2 + ax + 1 = 0$ have (1) two distinct real roots, (2) a pair of repeated real roots, or (3) no real roots?

5. Suppose the roots of the polynomial $p(x)$ are $x = \{-1, 1, 3\}$. What is one possible expression for $p(x)$?

Solutions

Calculus

1. Use the chain rule to compute: $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(e^{3t}\sin(5t)\right)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{3t}\sin(5t)\right) = \left(3e^{3t}\right)\sin(5t) + e^{3t}\left(5\cos(5t)\right)$$
$$= e^{3t}\left(3\sin(5t) + 5\cos(5t)\right)$$

2. Evaluate the integral: $\displaystyle\int_0^t \sin(6\tau)\,\mathrm{d}\tau$

$$\int_0^t \sin(6\tau)\,\mathrm{d}\tau = \left[-\frac{1}{6}\cos(6\tau)\right]_{\tau=0}^{\tau=t}$$
$$= -\frac{1}{6}\cos(6t) + \frac{1}{6}\cos(0)$$
$$= \frac{1 - \cos(6t)}{6}$$

3. Solve the differential equation: $y'(t) + 3y(t) = 0$, with the initial condition $y(0) = 2$.

This is an ordinary differential equation with constant coefficients, so the solution should be an exponential. Substituting in $y(t) = a \cdot e^{bt}$, we obtain:

$$abe^{bt} + 3ae^{bt} = 0$$

Therefore, $a(b+3)e^{bt} = 0$, and we conclude that $b = -3$. Since $y(0) = 2$, we conclude that $a = 2$. Therefore the solution is $y(t) = 2e^{-3t}$.

Another way to solve this differential equation is to use an integrating factor and the product rule. Multiply both sides by e^{3t} and obtain:

$$y'(t) + 3y(t) = 0$$
$$\implies \qquad e^{3t}y'(t) + 3e^{3t}y(t) = 0$$
$$\implies \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{3t}y(t)\right) = 0$$
$$\implies \qquad e^{3t}y(t) = C$$
$$\implies \qquad y(t) = Ce^{-3t}$$

And then we substitute $t = 0$ to conclude that $C = 2$ and the solution is $y(t) = 2e^{-3t}$.

Complex numbers

In the problems below, j is the imaginary unit; i.e. $j^2 = -1$.

1. Compute the real part $\mathrm{Re}(z)$ and the imaginary part $\mathrm{Im}(z)$ of the complex number $z = -2 + 3j$.

   The real part is -2 and the imaginary part is 3.

2. Compute the magnitude $|z|$ and argument $\arg(z)$ of $z = -1 + j$. For the argument, express your answer in both radians and degrees.

   The magnitude is $|-1 + j| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$. The argument is the angle that the point $(-1, 1)$ makes. This is $135°$, or $\frac{3\pi}{4}$ radians.

3. Draw the complex numbers $z_1 = 2 + 4j$ and $z_2 = -1 + j$ in the complex plane.

   These are the points $(2, 4)$ and $(-1, 1)$ on a standard Cartesian plane. The x-axis is the real part and the y-axis is the imaginary part.

4. Compute the real and imaginary parts of $z = 3e^{j\pi}$.

   Using Euler's formula: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$. In this case, we have:

   $$3e^{j\pi} = 3\left(\cos(\pi) + 3j\sin(\pi)\right)$$
   $$= 3\left(-1 + j \cdot 0\right)$$
   $$= -3$$

5. Compute $|e^{j\omega}|$, where ω is a real number.

   Again using Euler's formula:

   $$|e^{j\omega}| = \sqrt{\cos^2(\omega) + \sin^2(\omega)} = \sqrt{1} = 1$$

6. Compute $\arg(e^{j\omega})$, where ω is a real number. Express your answer in both radians and degrees.

   The real is $\cos(\omega)$ and the imaginary part is $\sin(\omega)$. So the angle is simply ω (radians), or $\frac{180\omega}{\pi}$ degrees.

7. Compute $\left|-2(-1 + 2j)(-4 - 3j)\right|$.

   We can evaluate the number directly and then compute its magnitude:

   $$-2(-1 + 2j)(-4 - 3j) = -2(4 - 8j + 3j + 6) = -20 + 10j$$

   Therefore $|-20 + 10j| = \sqrt{20^2 + 10^2} = \sqrt{500} = 10\sqrt{5}$.

   Alternatively, we can use the fact that the magnitude of the product of complex numbers is the product of the magnitudes:

   $$|-2(-1 + 2j)(-4 - 3j)| = |-2| \cdot |-1 + 2j| \cdot |-4 - 3j| = (2)(\sqrt{5})(5) = 10\sqrt{5}$$

8. Write $\dfrac{-3 + 2j}{3 - 4j}$ in the form $a + bj$ for some real numbers a and b.

   The technique is to multiply numerator and denominator by the complex conjugate of the denominator. This makes the denominator a real number.

   $$\frac{-3 + 2j}{3 - 4j} = \frac{(-3 + 2j)(3 + 4j)}{(3 - 4j)(3 + 4j)} = \frac{-17 - 6j}{25} = \frac{-17}{25} + \frac{-6}{25}j$$

Polynomials

1. Find all solutions to $x^2 - x + 4 = 0$

   Using the quadratic formula:

   $$x = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 4}}{2} = \frac{1}{2} \pm \frac{\sqrt{15}}{2} j$$

2. Find all solutions to $3x^2 + 2x + 1 = 0$.

   Using the quadratic formula:

   $$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 3 \cdot 1}}{2 \cdot 3} = -\frac{1}{3} \pm \frac{\sqrt{8}}{6} j = -\frac{1}{3} \pm \frac{\sqrt{2}}{3} j$$

3. Simplify the expression $\dfrac{x^2 - 2x - 3}{x^2 + 3x + 2}$ by canceling common factors.

   $$\frac{x^2 - 2x - 3}{x^2 + 3x + 2} = \frac{\cancel{(x+1)}(x-3)}{\cancel{(x+1)}(x+2)} = \frac{x-3}{x+2}$$

4. For what values of a (real number) does the equation $x^2 + ax + 1 = 0$ have (1) two distinct real roots, (2) a pair of repeated real roots, or (3) no real roots?

   The discriminant is $\Delta = a^2 - 4$. This is the piece under the square root in the quadratic formula. If $\Delta > 0$, we have two distinct real roots. If $\Delta = 0$, we have repeated real roots, and if $\Delta < 0$, we have a pair of complex conjugate roots (no real roots). This means that:

   $$\text{possible cases:} \begin{cases} a < -2 & \text{distinct real roots} \\ a = -2 & \text{repeated real roots} \\ -2 < a < 2 & \text{complex conjugate roots} \\ a = 2 & \text{repeated real roots} \\ a > 2 & \text{distinct real roots} \end{cases}$$

5. Suppose the roots of the polynomial $p(x)$ are $x = \{-1, 1, 3\}$. What is one possible expression for $p(x)$?

   If a is a root, that means that $(x - a)$ is a factor. So one possible choice for $p(x)$ would be:

   $$p(x) = (x + 1)(x - 1)(x - 3)$$

   There can also be a coefficient in front, so we can have $p(x) = k(x + 1)(x - 1)(x - 3)$ for any $k \neq 0$.

   Note: If the list of roots in the problem does not include multiplicity, we could even have something like $p(x) = k(x + 1)^m (x - 1)^n (x - 3)^q$ for positive integers m, n, q.