Extra worked circuit problems

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This document presents worked solutions to two circuit problems using the systematic method of writing component equations and eliminating currents via KCL.

- The first problem is a simple parallel RLC circuit.
- The second problem is a much more challenging circuit. This problem is more complicated than anything we would ever ask you to do by hand; the point is to demonstrate that the method still works even when the circuit is complicated.

Complete solutions are included after each problem.

1 RLC in parallel

Consider the following circuit:

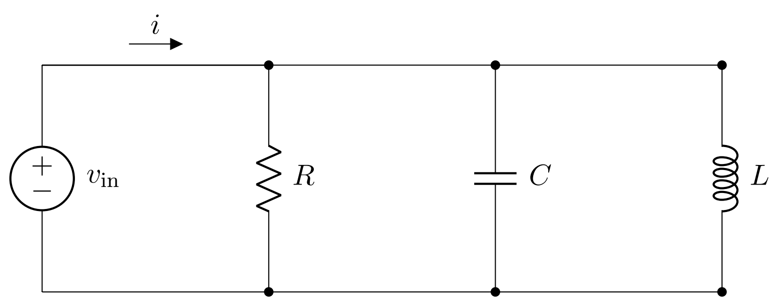

Figure 1: Parallel RLC circuit.

Find a differential equation that relates v_{in} to i.

Detailed solution on the following page.

Solution to parallel RLC problem

Let's start by labeling the unknown currents and adding a ground reference at the negative terminal of the voltage source.

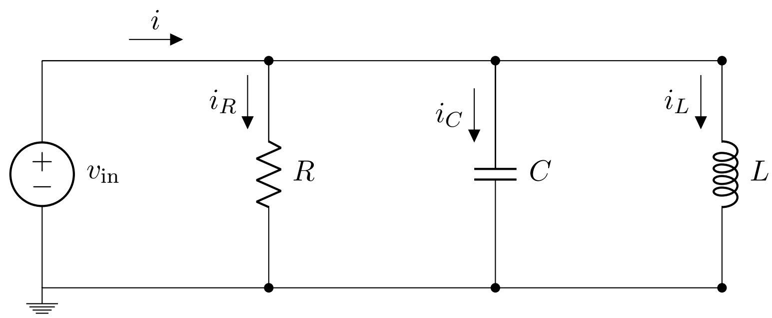

Figure 2: Parallel RLC circuit with currents labeled.

There are no nodes to label, since the entire top part of the circuit is at v_{in} and the entire bottom part is at ground. Let's write equations for each of the components, and then apply KCL at the top junction. We obtain:

$$v_{\text{in}} = Ri_R \tag{1a}$$
$$v_{\text{in}} = \tfrac{1}{C}\int i_C \, dt \tag{1b}$$
$$v_{\text{in}} = L\frac{di_L}{dt} \tag{1c}$$
$$i = i_R + i_L + i_C \tag{1d}$$

We have 4 equations in 5 unknowns: $\{v_{\text{in}}, i, i_R, i_L, i_C\}$, which is exactly what we want, because we would like to eliminate the three variables $\{i_R, i_L, i_C\}$ so that we are left with a single equation in the two variables $\{v_{\text{in}}, i\}$. We can isolate each of $\{i_R, i_L, i_C\}$ from (1a)–(1c) to obtain:

$$i_R = \tfrac{1}{R}v_{\text{in}} \tag{2a}$$
$$i_C = C\dot{v}_{\text{in}} \tag{2b}$$
$$i_L = \tfrac{1}{L}\int v_{\text{in}} \, dt \tag{2c}$$
$$i = i_R + i_L + i_C \tag{2d}$$

Substitute (2a)–(2c) into (2d) and obtain:

$$i = \tfrac{1}{R}v_{\text{in}} + C\dot{v}_{\text{in}} + \tfrac{1}{L}\int v_{\text{in}} \, dt \tag{3}$$

We can differentiate both sides to get rid of the integral and we obtain our final answer:

$$\tfrac{di}{dt} = C\ddot{v}_{\text{in}} + \tfrac{1}{R}\dot{v}_{\text{in}} + \tfrac{1}{L}v_{\text{in}} \tag{4}$$

2 A more complicated problem

Consider the following circuit diagram:

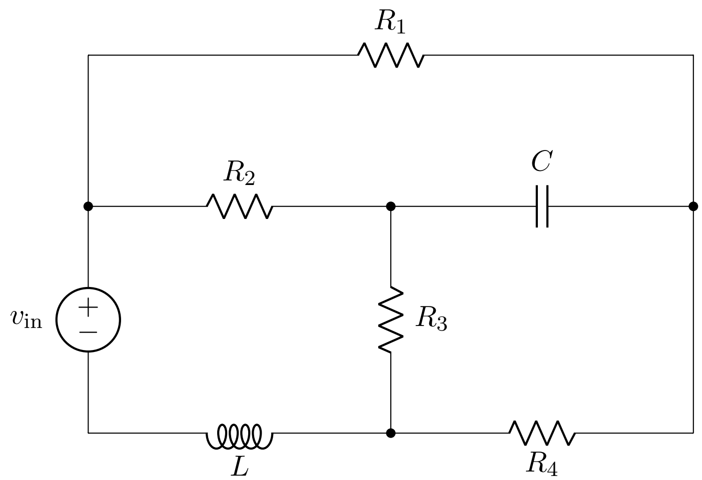

Figure 3: Circuit diagram

(a) What is the voltage drop across the capacitor as a function of v_{in}?

(b) What happens to the voltage drop across the capacitor when $R_1 = R_2 = R_3 = R_4$? Why?

Detailed solution in the following pages!

Solution to part (a)

First, redraw the diagram, labeling the unlabeled nodes and adding currents across each element. Let's also add a ground at the negative terminal of the voltage source. The current directions are arbitrary, but remember the convention that voltage drops in the direction of current flow.

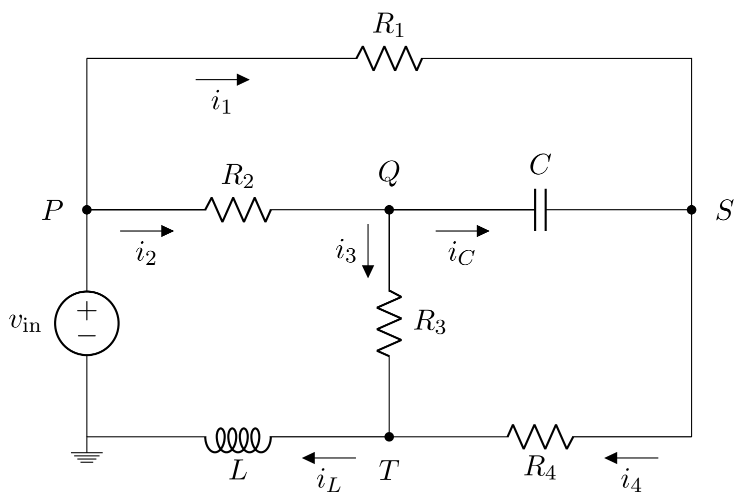

Figure 4: Circuit diagram with nodes and currents labeled

Next, let's write equations for each of the components, and then apply KCL at each junction. Since we are interested in the voltage across the capacitor, let's call that v_o (output voltage) and include it as one of our equations.

$$\begin{aligned}
\text{voltage source}: &\quad v_P = v_\mathrm{in} && (1a) \\
\text{component } R_1: &\quad v_P - v_S = R_1 i_1 && (1b) \\
\text{component } R_2: &\quad v_P - v_Q = R_2 i_2 && (1c) \\
\text{component } C: &\quad v_Q - v_S = \tfrac{1}{C}\textstyle\int i_C\,dt && (1d) \\
\text{component } R_3: &\quad v_Q - v_T = R_3 i_3 && (1e) \\
\text{component } R_4: &\quad v_S - v_T = R_4 i_4 && (1f) \\
\text{component } L: &\quad v_T = L\frac{di_L}{dt} && (1g) \\
\text{output definition}: &\quad v_Q - v_S = v_\mathrm{o} && (1h)
\end{aligned}$$

$$\begin{aligned}
\text{node } P: &\quad i_L = i_1 + i_2 && (2a) \\
\text{node } Q: &\quad i_2 = i_3 + i_C && (2b) \\
\text{node } S: &\quad i_4 = i_1 + i_C && (2c) \\
\text{node } T: &\quad i_L = i_3 + i_4 && (2d)
\end{aligned}$$

We have a total of 12 equations, and 12 variables: $\{v_P, v_Q, v_S, v_T, i_1, i_2, i_3, i_4, i_C, i_L, v_\mathrm{in}, v_\mathrm{o}\}$. So we actually have too many equations! Since our hope is to eliminate 10 variables and be left with just a single equation in $\{v_\mathrm{in}, v_o\}$, we should have 11 equations, not 12! The culprit here is that one of our KCL equations is redundant. (read: the four KCL equations are linearly dependent). To see why, notice that if we combine the first three KCL equations: (2a) + (2b) − (2c), we actually obtain (2d). Another way to say this is that the 4 KCL equations are *linearly dependent.* Since (2d) doesn't tell us anything new, we can ignore it.

We can now start eliminating variables and reducing the number of equations. The first thing to do is to eliminate v_P since $v_P = v_{\text{in}}$ from Eq. (1a) and to eliminate either v_Q or v_S from Eq. (1h), since we want to make sure our solution includes v_o. Let's eliminate v_Q by substituting $v_Q = v_S + v_o$ everywhere. After eliminating v_P and v_Q and also removing (2d) since it is redundant, our equations become:

$$\begin{aligned}
&\text{component } R_1: & v_{\text{in}} - v_S &= R_1 i_1 && \text{(3a)} \\
&\text{component } R_2: & v_{\text{in}} - v_S - v_o &= R_2 i_2 && \text{(3b)} \\
&\text{component } C: & v_o &= \tfrac{1}{C}\textstyle\int i_C\, dt && \text{(3c)} \\
&\text{component } R_3: & v_S + v_o - v_T &= R_3 i_3 && \text{(3d)} \\
&\text{component } R_4: & v_S - v_T &= R_4 i_4 && \text{(3e)} \\
&\text{component } L: & v_T &= L\frac{di_L}{dt} && \text{(3f)}
\end{aligned}$$

$$\begin{aligned}
&\text{node } P: & i_L &= i_1 + i_2 && \text{(4a)} \\
&\text{node } Q: & i_2 &= i_3 + i_C && \text{(4b)} \\
&\text{node } S: & i_4 &= i_1 + i_C && \text{(4c)}
\end{aligned}$$

We now have 9 equations in 10 variables: $\{v_S, v_T, i_1, i_2, i_3, i_4, i_C, i_L, v_{\text{in}}, v_{\text{o}}\}$. From here, there are many ways to proceed; we could pick any order we like to eliminate variables... My favorite way is to eliminate all the currents first. The easiest way to do this is to use the equations (3a)–(3f), since there is one equation for each different current we want to eliminate. Let's start by rewriting the component equations (3a)–(3f) to isolate each current.

$$\begin{aligned}
&\text{component } R_1: & i_1 &= \tfrac{1}{R_1}\left(v_{\text{in}} - v_S\right) && \text{(5a)} \\
&\text{component } R_2: & i_2 &= \tfrac{1}{R_2}\left(v_{\text{in}} - v_S - v_o\right) && \text{(5b)} \\
&\text{component } C: & i_C &= C\dot{v}_o && \text{(5c)} \\
&\text{component } R_3: & i_3 &= \tfrac{1}{R_3}\left(v_S + v_o - v_T\right) && \text{(5d)} \\
&\text{component } R_4: & i_4 &= \tfrac{1}{R_4}\left(v_S - v_T\right) && \text{(5e)} \\
&\text{component } L: & i_L &= \tfrac{1}{L}\textstyle\int v_T\, dt && \text{(5f)}
\end{aligned}$$

$$\begin{aligned}
&\text{node } P: & i_L &= i_1 + i_2 && \text{(6a)} \\
&\text{node } Q: & i_2 &= i_3 + i_C && \text{(6b)} \\
&\text{node } S: & i_4 &= i_1 + i_C && \text{(6c)}
\end{aligned}$$

Now eliminate the currents by substituting (5a)–(5f) into (6a)–(6c). We are left with:

$$\begin{aligned}
&\text{node } P: & \tfrac{1}{L}\textstyle\int v_T\, dt &= \tfrac{1}{R_1}\left(v_{\text{in}} - v_S\right) + \tfrac{1}{R_2}\left(v_{\text{in}} - v_S - v_o\right) && \text{(7a)} \\
&\text{node } Q: & \tfrac{1}{R_2}\left(v_{\text{in}} - v_S - v_o\right) &= \tfrac{1}{R_3}\left(v_S + v_o - v_T\right) + C\dot{v}_o && \text{(7b)} \\
&\text{node } S: & \tfrac{1}{R_4}\left(v_S - v_T\right) &= \tfrac{1}{R_1}\left(v_{\text{in}} - v_S\right) + C\dot{v}_o && \text{(7c)}
\end{aligned}$$

We're down to 3 equations in 4 variables: $\{v_S, v_T, v_{\text{in}}, v_o\}$. Our goal is to eliminate v_T and v_S, so let's look for ways to isolate them. Let's rearrange all equations to put v_S and v_T on the left-hand side and see what we get. We'll also differentiate (7a) to get rid of the integral.

$$\begin{aligned}
&\text{node } P: & \left(\tfrac{1}{R_1} + \tfrac{1}{R_2}\right)\dot{v}_S + \tfrac{1}{L}v_T &= \left(\tfrac{1}{R_1} + \tfrac{1}{R_2}\right)\dot{v}_{\text{in}} - \tfrac{1}{R_2}\dot{v}_o && \text{(8a)} \\
&\text{node } Q: & \left(\tfrac{1}{R_2} + \tfrac{1}{R_3}\right)v_S - \tfrac{1}{R_3}v_T &= \tfrac{1}{R_2}v_{\text{in}} - \left(\tfrac{1}{R_2} + \tfrac{1}{R_3}\right)v_o - C\dot{v}_o && \text{(8b)} \\
&\text{node } S: & \left(\tfrac{1}{R_1} + \tfrac{1}{R_4}\right)v_S - \tfrac{1}{R_4}v_T &= \tfrac{1}{R_1}v_{\text{in}} + C\dot{v}_o && \text{(8c)}
\end{aligned}$$

Equations (8b) and (8c) only involve v_S and v_T (no derivatives), so we can solve these two equations for v_S and v_T. The result is:

$$v_S = v_{\text{in}} - \frac{R_1\left(R_2+R_3\right)}{R_1R_3-R_2R_4}v_o - \frac{R_1R_2\left(R_3+R_4\right)}{R_1R_3-R_2R_4}C\dot{v}_o \tag{9a}$$

$$v_T = v_{\text{in}} - \frac{\left(R_2+R_3\right)\left(R_1+R_4\right)}{R_1R_3-R_2R_4}v_o - \frac{\left(R_1R_2R_3+R_1R_4R_3+R_2R_4R_3+R_1R_2R_4\right)}{R_1R_3-R_2R_4}C\dot{v}_o \tag{9b}$$

Note that (9a) and (9b) came from solving (8b) and (8c). So we should substitute our results into (8a); the only equation we haven't used yet. Substituting v_S and v_T from (9a) and (9b) into (8a), we obtain one equation in the variables $\{v_{\text{in}}, v_o\}$ (this is one long equation broken across three lines)

$$\begin{aligned}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)&\left(\dot{v}_{\text{in}} - \frac{R_1\left(R_2+R_3\right)}{R_1R_3-R_2R_4}\dot{v}_o - \frac{R_1R_2\left(R_3+R_4\right)}{R_1R_3-R_2R_4}C\ddot{v}_o\right)\\ &+\frac{1}{L}\left(v_{\text{in}} - \frac{\left(R_2+R_3\right)\left(R_1+R_4\right)}{R_1R_3-R_2R_4}v_o - \frac{\left(R_1R_2R_3+R_1R_4R_3+R_2R_4R_3+R_1R_2R_4\right)}{R_1R_3-R_2R_4}C\dot{v}_o\right)\\ &= \left(\frac{1}{R_1}+\frac{1}{R_2}\right)\dot{v}_{\text{in}} - \frac{1}{R_2}\dot{v}_o\end{aligned} \tag{10}$$

This is it! We can simplify this a bit. Notice that the $\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\dot{v}_{\text{in}}$ terms cancel from both sides, leaving us with (again, one equation broken across two lines):

$$\begin{aligned}&\frac{1}{R_2}\dot{v}_o - \left(\frac{1}{R_1}+\frac{1}{R_2}\right)\left(\frac{R_1\left(R_2+R_3\right)}{R_1R_3-R_2R_4}\dot{v}_o + \frac{R_1R_2\left(R_3+R_4\right)}{R_1R_3-R_2R_4}C\ddot{v}_o\right)\\ &+\frac{1}{L}\left(v_{\text{in}} - \frac{\left(R_2+R_3\right)\left(R_1+R_4\right)}{R_1R_3-R_2R_4}v_o - \frac{\left(R_1R_2R_3+R_1R_4R_3+R_2R_4R_3+R_1R_2R_4\right)}{R_1R_3-R_2R_4}C\dot{v}_o\right) = 0\end{aligned} \tag{11}$$

To make things a bit neater, we can collect the v_o terms on the left and the v_{in} term on the right and we obtain:

$$p\ddot{v}_o + q\dot{v}_o + rv_o = kv_{\text{in}}$$

where the constants are:

$$\begin{aligned}p &= LC(R_1+R_2)(R_3+R_4)\\ q &= L(R_1+R_2+R_3+R_4) + CR_1R_2R_3R_4\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right)\\ r &= (R_1+R_4)(R_2+R_3)\\ k &= (R_1R_3-R_2R_4)\end{aligned}$$

Solution to part (b)

Based on the solution of part (a), if we let $R = R_1 = R_2 = R_3 = R_4$, the ODE simplifies to:

$$LC\ddot{v}_o + \left(\tfrac{L}{R} + RC\right)\dot{v}_o + v_o = 0$$

Most importantly, the right-hand side is zero, and does not depend on v_{in}! So when the dust settles, $v_o \to 0$. There is eventually no voltage drop across the capacitor. But why?

One way to see this is to redraw the diagram in a more symmetric fashion. Here is the original diagram, and then the redrawn version:

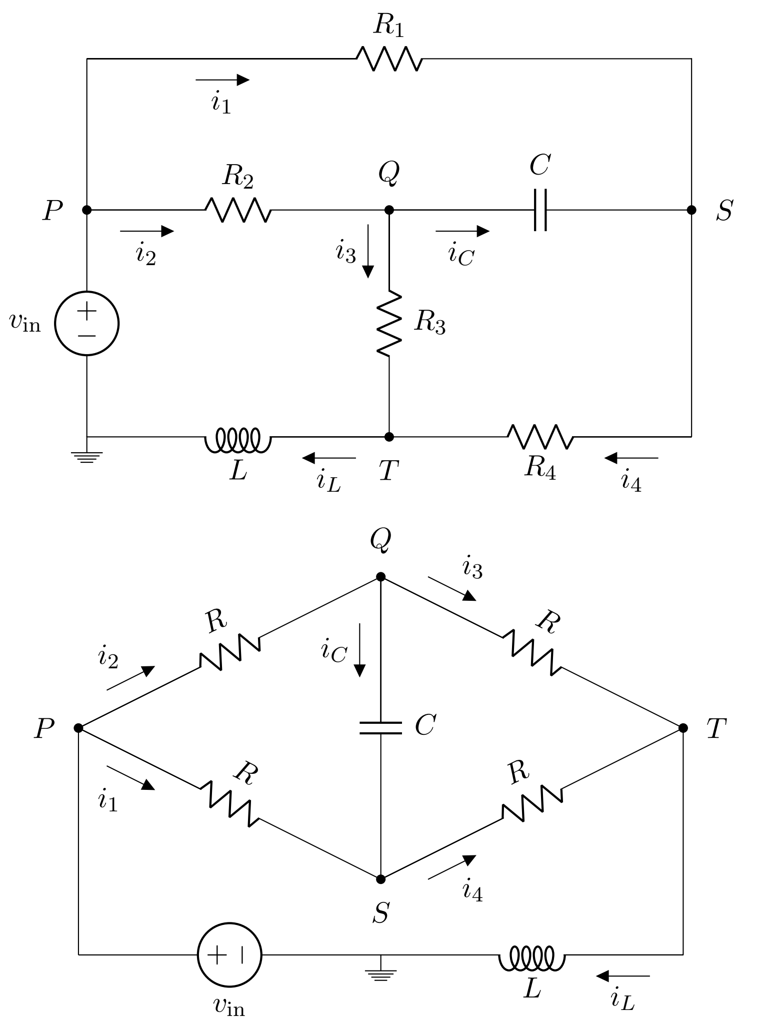

Figure 5: Circuit diagram drawn two different (but equivalent) ways.

When drawn in this different way, we can see that the top part of the circuit is perfectly symmetric. The paths $P \to Q \to T$ and $P \to S \to T$ are identical, so we would expect $v_S = v_Q$ and $i_C = 0$ by symmetry. It follows that $v_o = v_S - v_Q = 0$.