ECE 717 – Lecture 20 – LQR

We saw how to choose feedback control in order to change the closed-loop dynamics by placing poles in desirable locations (e.g. to achieve good transient / steady-state / tracking properties).

Another way to find a controller is to choose it via <u>optimization</u>. For example, if $x(t)$ is our tracking error at time t and $u(t)$ is control effort, we might want to make error small while exerting small control effort. One way (there are many) of doing this is using quadratic functions:

$$J_x = \int_0^T x(t)^T Q\, x(t)\, dt \qquad \text{(tracking error)}$$

$$J_u = \int_0^T u(t)^T R\, u(t)\, dt \qquad \text{(control effort)}$$

We choose $Q \geq 0$ (some states may not be penalized) and $R > 0$ (all inputs penalized). A typical choice is $R = \mu I$ with $\mu > 0$. Here, μ controls the relative cost of input effort. Solving the problem for different $\mu > 0$ allows us to trade-off J_x and J_u.

The LQR (Linear Quadratic Regulator):

choose $u(t)$ such that we minimize

$$J = \int_0^\infty \left(x(t)^T Q\, x(t) + u(t)^T R\, u(t) \right) dt$$

where $\dot{x}(t) = Ax(t) + Bu(t)$.

Note: this is the version we will study. It is called the "infinite-horizon, continuous-time, LQR problem". There is a similar treatment for discrete-time dynamics (integral becomes a sum) and also for finite-horizon (integral becomes $\int_0^T$ for some finite $T > 0$ instead of $\int_0^\infty$).

Note: We might expect the optimal (in the sense of minimizing J) control strategy to depend on the initial state $x(0)$. It turns out that the optimal J depends on $x(0)$, but the strategy that optimizes J does not depend on $x(0)$!

There are many ways to solve this problem (dynamic programming, calculus of variations, spectral factorization, ...) and many ways to interpret the solution. We will use an approach based on simple algebra and calculus.

But first, we need a result: the vector version of "completing the square"!

Completing the square in 1-D:

let $f(x,u) = qx^2 + 2sxu + ru^2$ where $r > 0$

If we want to minimize $f(x,u)$ with respect to u, we can "complete the square":

$$f(x,u) = qx^2 + r\left(u^2 + 2\frac{sx}{r}u\right)$$

$$= qx^2 + r\left(u^2 + 2\left(\frac{sx}{r}\right)u + \left(\frac{sx}{r}\right)^2\right) - r\left(\frac{sx}{r}\right)^2$$

$$= \underbrace{\left(q - \frac{s^2}{r}\right)x^2}_{\text{doesn't depend on } u} + \underbrace{r\left(u + \frac{sx}{r}\right)^2}_{\text{can be made zero (pick } u = -sx/r\text{.)}}$$

the optimal choice is $u_{opt} = -\frac{sx}{r}$ and the optimal value is $f(x, u_{opt}) = \left(q - \frac{s^2}{r}\right)x^2$.

<u>Note</u>: u_{opt} is linear in x, f_{opt} is quadratic in x.

We can do something similar if x, u are <u>vectors</u>.

<u>Note</u>: we can also achieve this result by taking the derivative: $\frac{d}{du} f(x,u) = 2sx + 2ru. = 0$

$$\Rightarrow u = -\frac{sx}{r}.$$

If $r < 0$ instead, setting $\frac{d}{du} f = 0$ finds the maximum instead (minimum is $-\infty$).

Completing the square for vectors:

let $f(x,u) = x^T Q x + 2x^T S u + u^T R u$ where $R > 0$.

We can complete the square, taking care and using fact that $R > 0$ means $R^{-1} > 0$ also:

$$
\begin{aligned}
f(x,u) &= x^T Q x + \left(u^T R u + u^T S^T x + x^T S u \right) \\
&= x^T Q x + \left(u + R^{-1} S^T x \right)^T R \left(u + R^{-1} S^T x \right) - x^T S R^{-1} S^T x \\
&= \underbrace{x^T \left(Q - S R^{-1} S^T \right) x}_{\text{only depends on } x} + \underbrace{\left(u + R^{-1} S^T x \right)^T R \left(u + R^{-1} S^T x \right)}_{\text{always } \geq 0 \text{ because } R > 0. \text{ Can be made zero if } u = -R^{-1} S^T x}
\end{aligned}
$$

Therefore, $u_{opt} = -R^{-1} S^T x$ (linear function of x)

and $f(x, u_{opt}) = x^T \left(Q - S R^{-1} S^T \right) x$ (quadratic function of x).

<u>Note</u>: taking derivatives (vector): $\frac{d}{du} f(x,u) = 2S^T x + 2Ru = 0$

$\Rightarrow u = -R^{-1} S^T x$.

Also, neat fact: $f(x,u) = \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$

and $\min_u f(x,u) = x^T \underbrace{\left(Q - S R^{-1} S^T \right)}_{\text{schur complement of } \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} !} x$

Back to LQR. We will solve it using a clever completion of squares. We start by introducing a matrix $P = P^T$:

$$J = \int_0^\infty \left(x^T Q x + u^T R u\right) dt$$

$$= x_0^T P x_0 + \int_0^\infty \Big(\underbrace{\tfrac{d}{dt}(x^T P x)}_{= \left[x^T P x\right]_0^\infty = -x_0^T P x_0 \text{ since } x(t) \xrightarrow{t\to\infty} 0.} + x^T Q x + u^T R u \Big) dt$$

Use the fact that $\dot{x} = Ax + Bu$ and $\frac{d}{dt}(x^T P x) = \dot{x}^T P x + x^T P \dot{x}$.

$$= x_0^T P x_0 + \int_0^\infty \Big(\underbrace{(Ax+Bu)^T Px + x^T P (Ax + Bu) + x^T Q x + u^T R u}_{\text{let's choose } u \text{ to minimize this! complete the square...}} \Big) dt.$$

$$= x_0^T P x_0 + \int_0^\infty \Big(x^T \underbrace{(A^T P + PA + Q)}_{\text{"}Q\text{"}} x + 2 x^T \underbrace{PB}_{\text{"}S\text{"}} u + u^T R u \Big) dt$$

$$= \underbrace{x_0^T P x_0}_{\substack{\text{doesn't depend}\\ \text{on } u(t) \text{ at all}}} + \int_0^\infty \Big[\underbrace{x^T (A^T P + PA + Q - PBR^{-1}B^T P) x}_{\text{can we make this zero?}} + \underbrace{(u + R^{-1}B^T P x)^T R (u + R^{-1}B^T P x)}_{\substack{\text{can always make zero by}\\ \text{setting } u = -R^{-1}B^T P x.}} \Big] dt$$

If we can find P such that $A^T P + PA + Q - PBR^{-1}B^T P = 0$ then the optimal control policy is given by: $u = -R^{-1}B^T P x$.

<u>Note</u>: this is a state feedback controller!

The equation $A^TP + PA + Q - PBR^{-1}B^TP = 0$ (*)
is called the "algebraic Riccati equation" (Matlab: "care")

You can think of it as the matrix version of a quadratic equation. There is no equivalent of the "quadratic formula" but solving an ARE is equivalent to finding eigenvalues, so it can be done efficiently.

<u>Theorem</u> Suppose $Q \geq 0$, $R > 0$ and (A,B) is stabilizable, and (A,Q) is detectable. Then we have:

1) the ARE has a unique solution that is symmetric and positive definite; $P = P^T > 0$.

2) $A - BR^{-1}B^TP$ is Hurwitz for this particular P.

Note that our optimal LQR controller was $u = -Kx$ where $K = R^{-1}B^TP$. So the theorem tells us that $(A-BK)$ is stable! i.e. if we use the LQR controller $u = -Kx$, we will have a stable closed loop and J will be minimized for any $x(0)$. $J_{min} = x(0)^T P x(0)$.

Stability can also be verified by manipulating the ARE:

letting $K = R^{-1}B^TP$, we have:

$$A^TP + PA + Q - PBR^{-1}B^TP = 0$$

$$\Rightarrow (A-BK)^TP + P(A \quad BK) + (Q + K^TRK) = 0$$

This is a Lyapunov equation in P. We know $P > 0$ is a solution. So if we want to prove $A-BK$ is stable:

$\Rightarrow$ let (x, λ) be an eigenpair of $A-BK$. So $(A-BK)x = \lambda x$. and $x \neq 0$. Then we have:

$$x^T\left[(A-BK)^TP + P(A-BK) + (Q+K^TRK)\right]x = 0$$

$$\Rightarrow (\lambda + \bar{\lambda})x^TPx + (x^TQx + u^TRu) = 0.$$

$$\Rightarrow 2\,Re(\lambda)\,x^TPx + (x^TQx + u^TRu) = 0$$

Since $x \neq 0$, $P > 0$, $x^TPx > 0$. We want to show that $Re(\lambda) < 0$. Clearly $Re(\lambda) > 0$ can't happen since $x^TPx > 0$, $x^TQx \geq 0$, $u^TRu \geq 0$. What if $Re(\lambda) = 0$? Then by detectability of (Q, A) and the PBH test for detectability, we must have $Qx \neq 0$. Let $Q = F^TF$ clearly $Fx \neq 0$ (otherwise $F^TFx = 0$, contradiction). Therefore $\|Fx\|^2 \neq 0$.

$\Rightarrow x^TF^TFx = x^TQx \neq 0$. So $x^TQx > 0$. i.e. $Re(\lambda) = 0$ is not possible. So $Re(\lambda) < 0 \Rightarrow A-BK$ is stable. ∎