ECE 717

Homework 6: Observers and optimal control

due: Sunday December 8, 2019

1. State observer. Consider the plant:

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

Compute a state observer for this plant such that the estimation error decays at a rate e^{-10t}. Write out the dynamics of the observer and draw a block diagram.

2. LQR with cross-terms. Find the optimal control policy under the dynamics $\dot{x} = Ax + Bu$ that optimizes the functional:

$$J(x_0) = \int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^\mathsf{T} \begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \mathrm{d}t$$

where $R \succ 0$ and $\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \succeq 0$, so J is always nonnegative.

Note that the version we solved in class was with $S = 0$. Following a similar derivation, you should obtain a similar solution but with a slightly different Algebraic Riccati Equation.

3. Algebraic Riccati Equations. We will study the simplified ARE:

$$A^\mathsf{T} X + XA + Q + XRX = 0 \tag{1}$$

where $Q, R \in \mathbb{R}^{n\times n}$ are symmetric matrices, and we want to find a solution $X \in \mathbb{R}^{n\times n}$.

a) Define the matrix $H = \begin{bmatrix} A & R \\ -Q & -A^\mathsf{T} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$. This is called the *Hamiltonian* matrix. Prove that we can rewrite the ARE (1) as $\begin{bmatrix} X & -I \end{bmatrix} H \begin{bmatrix} I \\ X \end{bmatrix} = 0$.

b) Define the matrix $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ where the blocks of J are the same sizes as the blocks of H. Show that $J^{-1}HJ = -H^\mathsf{T}$, and use this fact to prove that if λ is an eigenvalue of H, then so is $-\bar{\lambda}$. In other words, if H has no eigenvalues on the imaginary axis, then exactly n of them are stable and the other n are unstable.

c) Suppose we can find three matrices $X_1, X_2, M \in \mathbb{R}^{n\times n}$ such that X_1 is invertible and the matrices satisfy $H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} M$. Prove that $X = X_2 X_1^{-1}$ is a solution to the ARE (1).