ECE 717

Homework 5: State feedback

due: Monday November 25, 2019

1. Moving eigenvalues using feedback. Consider the LTI system:

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

a) Find a state-feedback gain matrix K such that $A + BK$ has eigenvalues at -1 and -2. Do this by directly computing eigenvalues.

b) Repeat the previous exercise, but this time solve it by transforming the system to controllable canonical form, finding K in those coordinates, and then transforming back to the original coordinates.

SOLUTION:

a) Suppose $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. Then we have:

$$A + BK = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 2 + k_1 & 1 + k_2 \\ -1 + 2k_1 & 1 + 2k_2 \end{bmatrix}$$

Since we want the eigenvalues at -1 and -2, the characteristic polynomial should be equal to $(\lambda + 1)(\lambda + 2) = \lambda^2 + 3\lambda + 2$. Computing this for the matrix above,

$$\begin{aligned} \lambda^2 + 3\lambda + 2 &= (\lambda - 2 - k_1)(\lambda - 1 - 2k_2) - (k_1 + 1)(2k_1 - 1) \\ &= \lambda^2 + (-3 - k_1 - k_2)\lambda + (3 + 5k_2 - k_1) \end{aligned}$$

Equating coefficients, we obtain the system:

$$\begin{aligned} -k_1 - 2k_2 - 3 &= 3 \\ -k_1 + 5k_2 + 3 &= 2 \end{aligned}$$

Solving, yields $k_1 = -4$ and $k_2 = -1$. Therefore $K = \begin{bmatrix} -4 & -1 \end{bmatrix}$.

b) The transformation to CCF is $T_{CCF} = P\, P_{CCF}^{-1} = \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -5 & 2 \end{bmatrix}$. Using this to transform the system, we obtain:

$$\left[\begin{array}{c|c} A_{CCF} & B_{CCF} \\ \hline C_{CCF} & D_{CCF} \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -3 & 3 & 1 \\ \hline -4 & 3 & 0 \end{array}\right]$$

Note that $a_0 = 3$ and $a_1 = -3$. Our desired characteristic polynomial is $\lambda^2 + 3\lambda + 2$, so $\alpha_0 = 2$ and $\alpha_1 = 3$. Therefore, we should use the controller $K_{CCF} = \begin{bmatrix} (a_0 - \alpha_0) & (a_1 - \alpha_1) \end{bmatrix} = \begin{bmatrix} 1 & -6 \end{bmatrix}$. Transforming back to the original coordinates,

$$K = K_{CCF} T_{CCF}^{-1} = \begin{bmatrix} 1 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -5 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -4 & -1 \end{bmatrix}$$

2. Stability via the controllability Gramian. Consider the LTI system: $\dot{x}(t) = Ax(t) + Bu(t)$ where (A, B) is controllable.

a) Prove that A is Hurwitz if and only if there is a solution $V \succ 0$ to the Lyapunov equation

$$AV + VA^\mathsf{T} + BB^\mathsf{T} = 0.$$

Hint: This problem is similar to Problem 2 in Homework 4.

b) Prove that A is Hurwitz if and only if there is a solution $V \succ 0$ to the equation

$$AV + VA^\mathsf{T} + BB^\mathsf{T} \preceq 0$$

In other words, we don't have to solve the Lyapunov equation exactly (equals zero) in order to prove stability. It's enough to solve so the left-hand side is negative semidefinite.
Hint: rearrange the inequality so it becomes an equality, and use Part a.

c) Define the controllability Gramian at time T to be:

$$W_T = \int_0^T e^{-A\tau} BB^\mathsf{T} e^{-A^\mathsf{T}\tau} \,\mathrm{d}\tau$$

Note that W_T is defined here with negative signs in the exponents. It turns out that if (A, B) is controllable, we have $W_T \succ 0$ for all $T > 0$ (proof is the same as with the standard Gramian definition). Prove that the following feedback law, for any $T > 0$, stabilizes the system.

$$u(t) = -B^\mathsf{T}(W_T)^{-1}x(t)$$

Hint: What is the feedback gain K matrix here? You're being asked to prove that $A + BK$ is Hurwitz. Make use of the result from Part b with $V = W_T$.

SOLUTION:

a) (if direction) Suppose $V \succ 0$ is a solution to the Lyapunov equation. Let (λ, w) be an eigenvalue of A and one of its associated left eigenvectors. Multiplying the Lyapunov equation on the left and right by w^* and w respectively, we obtain:

$$\begin{aligned} & w^*\left(AV + VA^\mathsf{T} + BB^\mathsf{T}\right)w = 0 \\ \implies \quad & (\lambda + \bar{\lambda})w^*Vw + \|B^\mathsf{T}w\|^2 = 0 \\ \implies \quad & 2\,\mathrm{Re}(\lambda)w^*Vw + \|B^\mathsf{T}w\|^2 = 0 \end{aligned}$$

Since $V \succ 0$ and $w \neq 0$, we have $w^*Vw > 0$. Moreover, we must have $B^\mathsf{T}w \neq 0$ because (A, B) is controllable (PBH test). Therefore, $\|B^\mathsf{T}w\| > 0$. We conclude that $\mathrm{Re}(\lambda) < 0$. This holds for any eigenvalue of A, therefore A is Hurwitz.
(only if direction) Conversely, suppose A is Hurwitz. Consider the matrix:

$$V = \int_0^\infty e^{A\tau} BB^\mathsf{T} e^{A^\mathsf{T}\tau} \,\mathrm{d}\tau$$

This integral is convergent because A is Hurwitz. It is straightforward to verify that V satisfies the Lyapunov equation. It remains to verify that $V \succ 0$. Suppose there exists some $w \neq 0$ such that $w^*Vw \leq 0$. Since

$$\begin{aligned} w^*Vw &= \int_0^\infty w^* e^{A\tau} BB^\mathsf{T} e^{A^\mathsf{T}\tau} w \,\mathrm{d}\tau \\ &= \int_0^\infty \left\| B^\mathsf{T} e^{A^\mathsf{T}\tau} w \right\|^2 \mathrm{d}\tau \end{aligned}$$

then it must be the case that $B^\mathsf{T} e^{A^\mathsf{T}\tau} w = 0$ for all $\tau \geq 0$. This implies that all derivatives with respect to τ are zero as well. Evaluating these derivatives at $\tau = 0$, we conclude that

$$w^* B = w^* AB = w^* A^2 B = \cdots = 0$$

Therefore $w^* P = 0$, where P is the controllability matrix. This is a contradiction because (A, B) is controllable, so the controllability matrix must have full row rank. We deduce that $P \succ 0$, as required.

b) (only if direction) Suppose A is Hurwitz. By Part a, there exists a $V \succ 0$ such that $AV + VA^\mathsf{T} + BB^\mathsf{T} = 0$. This same V also a solution to $AV + VA^\mathsf{T} + BB^\mathsf{T} \preceq 0$.

(if direction) Suppose $V \succ 0$ is a solution to $AV + VA^\mathsf{T} + BB^\mathsf{T} \preceq 0$. Put another way, if we define $R = AV + VA^\mathsf{T} + BB^\mathsf{T}$, then we have $R \preceq 0$. This means we can find a factorization $R = -SS^\mathsf{T}$ for some choice of S, e.g. by taking an eigenvalue decomposition. Rearranging the equation, we obtain:

$$\begin{aligned} && AV + VA^\mathsf{T} + BB^\mathsf{T} &= -SS^\mathsf{T} \\ \implies && AV + VA^\mathsf{T} + \begin{bmatrix} B & S \end{bmatrix} \begin{bmatrix} B & S \end{bmatrix}^\mathsf{T} &= 0 \end{aligned}$$

We have turned the Lyapunov inequality into an inequality. We can apply the result from Part a once again; note that (A, B) is controllable, therefore $(A, \begin{bmatrix} B & S \end{bmatrix})$ is also controllable by the PBH test. So A is Hurwitz, as required.

c) Since (A, B) is controllable, so is $(A + BK, B)$ for any K. We need to verify that $A + BK$ is Hurwitz when $K = -B^\mathsf{T} W_T^{-1}$. By applying the result from Part b, it suffices to find a solution to the Lyapunov equation

$$(A + BK)V + V(A + BK)^\mathsf{T} + BB^\mathsf{T} \preceq 0$$

Substituting our definition for K, we obtain:

$$(A - BB^\mathsf{T} W_T^{-1})V + V(A - BB^\mathsf{T} W_T^{-1})^\mathsf{T} + BB^\mathsf{T} \preceq 0$$

Let's try $V = W_T$ so the inverse cancels and so do several terms with BB^T and we get:

$$AW_T + W_T A^\mathsf{T} - BB^\mathsf{T} \preceq 0$$

Here is where the negative signs in the exponent come in. Rearranging:

$$(-A)W_T + W_T(-A)^\mathsf{T} + BB^\mathsf{T} \succeq 0$$

Substituting the definition for W_T, we obtain:

$$\begin{aligned} &(-A)W_T + W_T(-A)^\mathsf{T} + BB^\mathsf{T} \\ &\qquad = \int_0^T \left((-A)e^{-A\tau} BB^\mathsf{T} e^{-A^\mathsf{T}\tau} + e^{-A\tau} BB^\mathsf{T} e^{-A^\mathsf{T}\tau}(-A)^\mathsf{T} \right) \mathrm{d}\tau + BB^\mathsf{T} \\ &\qquad = \int_0^T \frac{\mathrm{d}}{\mathrm{d}\tau} \left(e^{-A\tau} BB^\mathsf{T} e^{-A^\mathsf{T}\tau} \right) \mathrm{d}\tau + BB^\mathsf{T} \\ &\qquad = \left(e^{-AT} BB^\mathsf{T} e^{-A^\mathsf{T}T} - BB^\mathsf{T} \right) + BB^\mathsf{T} \\ &\qquad = e^{-AT} BB^\mathsf{T} e^{-A^\mathsf{T}T} \\ &\qquad \succeq 0 \end{aligned}$$

The last line follows from the fact that $x^\mathsf{T}\left(e^{-AT}BB^\mathsf{T}e^{-A^\mathsf{T}T}\right)x = \left\|B^\mathsf{T}e^{-A^\mathsf{T}T}x\right\|^2 \geq 0$.

Note that we must be careful here. The equation $AW_T + W_TA^\mathsf{T} - BB^\mathsf{T} \preceq 0$ *looks* like a Lyapunov equation, but remember A itself is not Hurwitz... so none of our existing results apply! Indeed, we can't even let $T \to \infty$, because e^{-At} doesn't go to zero as $t \to \infty$.

3. Shaping the transient response. Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -680 & -176 & -86 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 100 & 20 & 10 & 0 \end{bmatrix} x$$

a) Determine the desired eigenvalues for a generic second-order system to obtain 2% overshoot and 2-s settling time.

b) Design a state feedback control law for the given system to achieve the transient requirements in part (a). Compare open- and closed-loop responses to a step input.

c) Design a state feedback control law using integral control for the given system to achieve the transient requirements in part (a) and a steady-state output value of 1. Place the poles using the 5$^{\text{th}}$ order ITAE polynomial: $s^5 + 2.8\omega_n s^4 + 5\omega_n^2 s^3 + 5.5\omega_n^3 s^2 + 3.4\omega_n^4 s + \omega_n^5$. Compare the open- and closed-loop responses to a step input.

SOLUTION:

a) Given a generic second order system: $G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

percent overshoot is given by $\exp\left(\frac{-\pi\xi}{\sqrt{1-\xi^2}}\right)$ and settling time is approximately $\frac{4}{\xi\omega_n}$. Solving percent overshoot yields:

$$\log(0.02) = \frac{-\pi\xi}{\sqrt{1-\xi^2}} \qquad \Longrightarrow \qquad \xi \approx 0.7797$$

Substituting into the settling time equation (equal to 2 seconds) and solving for ω_n yields $\omega_n \approx 2.565$. The poles of the transfer function are therefore located at:

$$\lambda = -\xi\omega_n \pm i\,\omega_n\sqrt{1-\xi^2} = -2 \pm 1.606i$$

b) The current characteristic polynomial is $s^4 + 6s^3 + 86s^2 + 176s + 680$, which has roots at $-1 \pm 3i$ and $-2 \pm 8i$. Let's place two of the poles at $-2 \pm 1.606i$ as in part (a), and the other two poles around 10 times farther left, so let's say at -20. Therefore we want our characteristic polynomial to be

$$(s + 2 + 1.606i)(s + 2 - 1.606i)(s + 20)^2 = s^4 + 44s^3 + 566.58s^2 + 1863.2s + 2631.7$$

Therefore, in controller canonical form coordinates, we need to shift by:

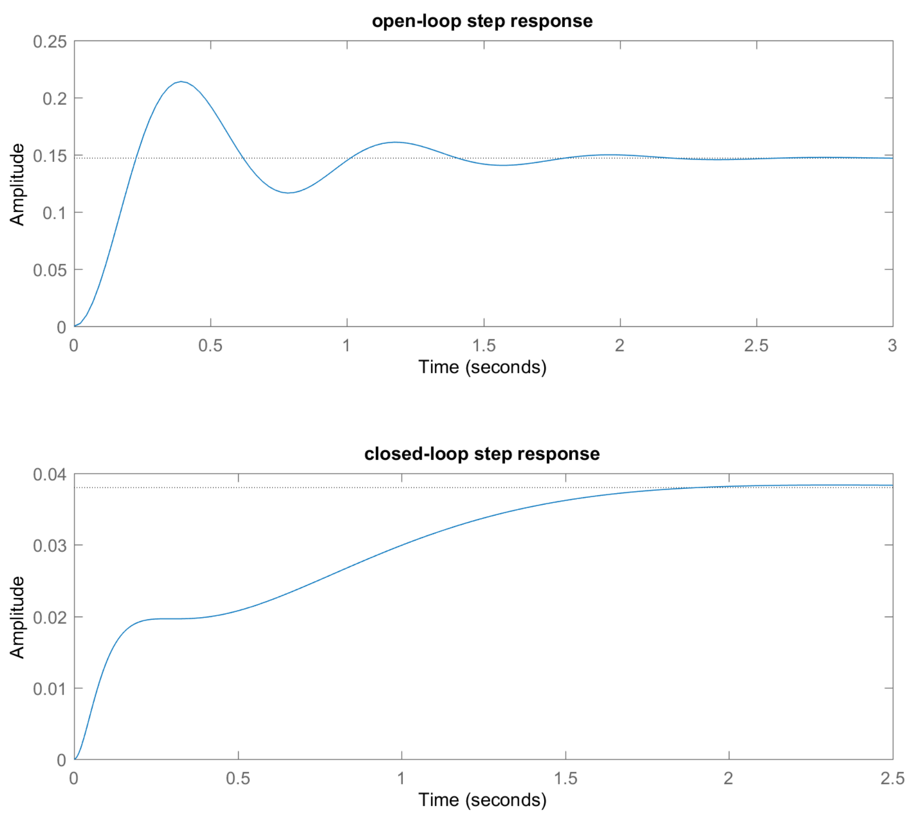
$$\begin{aligned} K_{CCF} &= \begin{bmatrix} (a_0 - \alpha_0) & (a_1 - \alpha_1) & (a_2 - \alpha_2) & (a_3 - \alpha_3) \end{bmatrix} \\ &= \begin{bmatrix} 680 - 2631.7 & 176 - 1863.2 & 86 - 566.58 & 6 - 44 \end{bmatrix} \\ &= \begin{bmatrix} -1951.7 & -1687.2 & -480.58 & -38 \end{bmatrix} \end{aligned}$$

The resulting closed-loop matrix is given by:

$$A_{CCF} + B_{CCF}K_{CCF} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2631.7 & -1863.2 & -566.58 & -44 \end{bmatrix}$$

Here is code and plots comparing the step responses of the two systems:

```
s = tf('s');
G1 = ss( [0 1 0 0; 0 0 1 0; 0 0 0 1; -680 -176 -86 -6],...
        [0;0;0;1], [100 20 10 0], 0 );
G2 = ss( [0 1 0 0; 0 0 1 0; 0 0 0 1; -2631.7 -1863.2 -566.58 -44],...
        [0;0;0;1], [100 20 10 0], 0 );
figure(1)
subplot(211); step(G1); title('open-loop step response')
subplot(212); step(G2); title('closed-loop step response')
```



c) Using the ITAE polynomial of fifth order, which is:

$$s^5 + 2.8\omega_n s^4 + 5\omega_n^2 s^3 + 5.5\omega_n^3 s^2 + 3.4\omega_n^4 s + \omega_n^5$$

We have $\omega_n = 2.565$ from part (a), therefore, the polynomial evaluates to:

$$s^5 + 7.182s^4 + 32.90s^3 + 92.81s^2 + 147.17s + 111.03$$

The closed-loop map using integral action with some (augmented) controller K is given by:

$$\left[\begin{array}{cc|c} A+BK & Bk_i & 0 \\ -C & 0 & 1 \\ \hline C & 0 & 0 \end{array}\right] = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ k_1-680 & k_2-176 & k_3-86 & k_4-6 & k_i & 0 \\ -100 & -20 & -10 & 0 & 0 & 1 \\ \hline 100 & 20 & 10 & 0 & 0 & 0 \end{array}\right]$$

The characteristic polynomial of this system is:

$$s^5 + (6-k_4)s^4 + (86-k_3)s^3 + (176-k_2+10k_5)s^2 + (680-k_1+20k_5)s + 100k_5$$

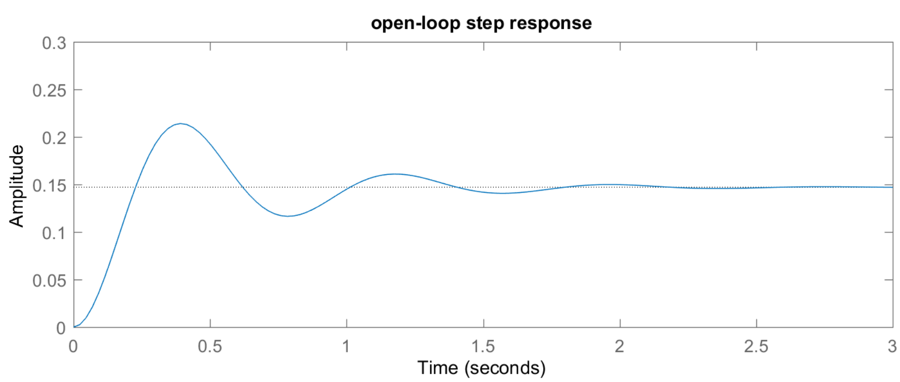
Comparing coefficients with the ITAE polynomial above, we conclude that:

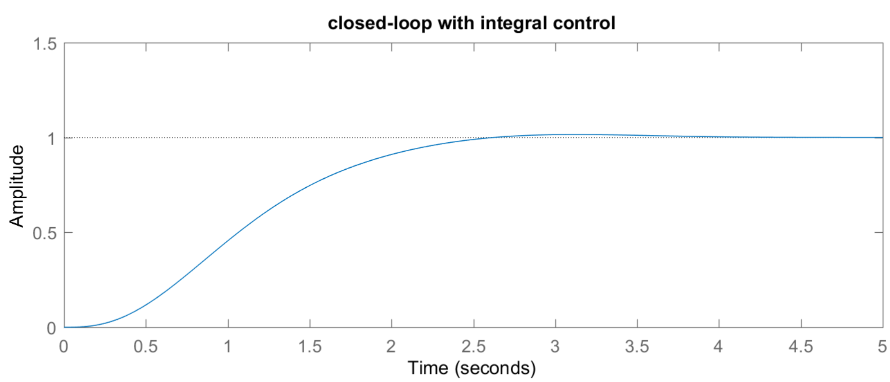
$$K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_i \end{bmatrix} = \begin{bmatrix} 555 & 94.29 & 53.1 & -1.182 & 1.11 \end{bmatrix}$$

So, more specifically, the control law is given by:

$$u(t) = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} x(t) + k_i \int_0^t (r(\tau) - y(\tau))\, \mathrm{d}\tau$$

The plot (similar code to before) is given below:





4. **Closed-loop stability using transfer functions.** Consider a system with transfer function

$$G(s) = \frac{(s-1)(s+2)}{(s+1)(s-2)(s+3)}.$$

Is it possible to make the closed-loop transfer function

$$\widehat{G}(s) = \frac{1}{s+3}$$

using state-feedback? Is the resulting system BIBO stable? Asymptotically stable?

SOLUTION: Yes, it is possible to make the closed-loop transfer function $\widehat{G}(s) = \frac{1}{s+3}$. For example, if we place $K(s)$ in the feedforward path, the closed-loop transfer function is $\frac{KG}{1-KG}$. In this case, picking $K(s) = \frac{(s+1)(s+3)(s-2)}{(s-1)(s+2)(s+4)}$ does the job. If, on the other hand, we place $K(s)$ in the feedback path, the closed-loop transfer function is $\frac{G}{1-KG}$. In this case, picking $K(s) = -\frac{2s(s+3)}{(s+2)(s-1)}$ does the job. Since the closed-loop transfer function is $\frac{1}{s+3}$, it is BIBO (input-output) stable by definition.

To see whether either possibility leads to an internally stable system, we can compute the realization of the feedback interconnection using the formulas seen in class. First, we use Matlab to compute minimal realizations for G and K. We'll demonstrate the second case (controller in feedback path), but you can do something similar for the feedforward path and come to a similar conclusion.

$$G(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] = \left[\begin{array}{ccc|c} -2 & 2.5 & 1.5 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0.5 & 0.25 & -0.25 & 0 \end{array}\right]$$

$$K(s) = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 2 & 4 \\ 1 & 0 & 0 \\ \hline -1 & -1 & -2 \end{array}\right]$$

The only thing that matters when considering internal stability is the eigenvalues of the closed-loop A-matrix. From the class notes (with $D_1 = 0$), this is:

$$A_{\text{cl}} = \begin{bmatrix} A_1 + B_1 D_2 C_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix} = \begin{bmatrix} -4 & 1.5 & 2.5 & -2 & -2 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 2 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Eigenvalues of this matrix are $\{-3, -2, 1, -2, 1\}$. Since at least one eigenvalue is unstable, the system is not asymptotically stable. Now, of course, since the closed-loop transfer function is $\frac{1}{s+3}$, only the -3 eigenvalue is controllable and observable. We could, in principle, reduce the realization. But the controller and plant are physically separate; these states are *physical states* that can't simply be canceled out!