ECE 717

Homework 5: State feedback

due: Monday November 25, 2019

1. Moving eigenvalues using feedback. Consider the LTI system:

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

a) Find a state-feedback gain matrix K such that $A + BK$ has eigenvalues at -1 and -2. Do this by directly computing eigenvalues.

b) Repeat the previous exercise, but this time solve it by transforming the system to controllable canonical form, finding K in those coordinates, and then transforming back to the original coordinates.

2. Stability via the controllability Gramian. Consider the LTI system: $\dot{x}(t) = Ax(t) + Bu(t)$ where (A, B) is controllable.

a) Prove that A is Hurwitz if and only if there is a solution $V \succ 0$ to the Lyapunov equation

$$AV + VA^{\mathsf{T}} + BB^{\mathsf{T}} = 0.$$

Hint: This problem is similar to Problem 2 in Homework 4.

b) Prove that A is Hurwitz if and only if there is a solution $V \succ 0$ to the equation

$$AV + VA^{\mathsf{T}} + BB^{\mathsf{T}} \preceq 0$$

In other words, we don't have to solve the Lyapunov equation exactly (equals zero) in order to prove stability. It's enough to solve so the left-hand side is negative semidefinite.
Hint: rearrange the inequality so it becomes an equality, and use Part a.

c) Define the controllability Gramian at time T to be:

$$W_T = \int_0^T e^{-A\tau} BB^{\mathsf{T}} e^{-A^{\mathsf{T}}\tau} \, \mathrm{d}\tau$$

Note that W_T is defined here with negative signs in the exponents. It turns out that if (A, B) is controllable, we have $W_T \succ 0$ for all $T > 0$ (proof is the same as with the standard Gramian definition). Prove that the following feedback law, for any $T > 0$, stabilizes the system.

$$u(t) = -B^{\mathsf{T}}(W_T)^{-1}x(t)$$

Hint: What is the feedback gain K matrix here? You're being asked to prove that $A + BK$ is Hurwitz. Make use of the result from Part b with $V = W_T$.

3. Shaping the transient response. Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -680 & -176 & -86 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 100 & 20 & 10 & 0 \end{bmatrix} x$$

a) Determine the desired eigenvalues for a generic second-order system to obtain 2% overshoot and 2-s settling time.

b) Design a state feedback control law for the given system to achieve the transient requirements in part (a). Compare open- and closed-loop responses to a step input.

c) Design a state feedback control law using integral control for the given system to achieve the transient requirements in part (a) and a steady-state output value of 1. Place the poles using the 5$^{\text{th}}$ order ITAE polynomial: $s^5 + 2.8\omega_n s^4 + 5\omega_n^2 s^3 + 5.5\omega_n^3 s^2 + 3.4\omega_n^4 s + \omega_n^5$. Compare the open- and closed-loop responses to a step input.

4. Closed-loop stability using transfer functions. Consider a system with transfer function

$$G(s) = \frac{(s-1)(s+2)}{(s+1)(s-2)(s+3)}.$$

Is it possible to make the closed-loop transfer function

$$\widehat{G}(s) = \frac{1}{s+3}$$

using state-feedback? Is the resulting system BIBO stable? Asymptotically stable?