ECE 717
Homework 4: Stability

due: Wednesday October 30, 2019

1. Warm-up. Consider the linear system $\dot{x} = Ax$. For each case, examine the eigenvalues to determine whether the system is (1) asymptotically stable, (2) marginally stable, or (3) unstable.

a) $A = \begin{bmatrix} 0 & 1 \\ -14 & -4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 0 & 1 \\ -14 & 4 \end{bmatrix}$ **c)** $A = \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix}$ **d)** $A = \begin{bmatrix} 0 & 1 \\ -14 & 0 \end{bmatrix}$

SOLUTION:

a) Eigenvalues are $-2 \pm 3.16228i$. Therefore, the system is asymptotically stable (negative real parts).

b) Eigenvalues are $2 \pm 3.16228i$. Therefore, the system is unstable (at least one eigenvalue has positive real part).

c) Eigenvalues are 0 and -4. Therefore, the system is marginally stable (all eigenvalues have nonpositive real part and one of them has zero real part).

d) Eigenvalues are $\pm 3.74166i$. Therefore, the system is marginally stable (eigenvalues are distinct and have zero real part).

2. A more general Lyapunov result. The Lyapunov stability result states that for any $Q = Q^\mathsf{T} \succ 0$, the equation $A^\mathsf{T}P + PA + Q = 0$ has a solution $P = P^\mathsf{T} \succ 0$ if and only if A is Hurwitz (all eigenvalues of A have negative real part). Using a similar proof technique, show the following result:

Suppose (C, A) is observable. Prove that the equation $A^\mathsf{T}W + WA + C^\mathsf{T}C = 0$ has a solution $W = W^\mathsf{T} \succ 0$ if and only if A is Hurwitz.

Note: this result is more general than the one we saw in class, because typically we will have $C^\mathsf{T}C \succeq 0$ but not $C^\mathsf{T}C \succ 0$. Be sure to prove both directions (if *and* only if).

SOLUTION:

($\Longrightarrow$) Suppose $W = W^\mathsf{T} \succ 0$ and $A^\mathsf{T}W + WA + C^\mathsf{T}C = 0$ and (C, A) is controllable. Let (λ, x) be an eigenpair of A. Then we have $Ax = \lambda x$. Multiplying both sides of the equation by x^* and x:

$$
\begin{aligned}
& x^* \left(A^\mathsf{T}W + WA + C^\mathsf{T}C \right) x = 0 \\
\Longrightarrow \quad & (\lambda + \bar{\lambda}) x^* W x + (Cx)^*(Cx) = 0 \\
\Longleftrightarrow \quad & 2\,\mathrm{Re}(\lambda)(x^* W x) + \|Cx\|^2 = 0
\end{aligned}
$$

By the PBH test, since $Ax = \lambda x$, we must have $Cx \neq 0$. Therefore, $\|Cx\| \neq 0$ and $\|Cx\|^2 > 0$. Since $W \succ 0$, we also have $x^* W x > 0$. Therefore, $\mathrm{Re}(\lambda) < 0$, as required.

($\Longleftarrow$) Suppose A is Hurwitz. Therefore $e^{At} \to 0$ as $t \to \infty$. Using a similar construction to what we used in class, let:

$$
W = \int_0^\infty e^{A^\mathsf{T}t} C^\mathsf{T} C e^{At} \, \mathrm{d}t
$$

It's clear that $W = W^\mathsf{T}$ and that W satisfies the matrix equation. To prove that $W \succ 0$, multiply left and right by an arbitrary nonzero vector $v \in \mathbb{R}^n$. We obtain:

$$v^\mathsf{T} W v = \int_0^\infty \left\| C e^{At} v \right\|^2 \mathrm{d}t$$

Suppose (proof by contradiction) that $\left\| C e^{At} v \right\| = 0$ for all t. Then setting derivatives equal to zero, we conclude that $Cv = 0$, $CAv = 0$, $CA^2 v = 0$, and so on. This means that $Qv = 0$, where Q is the observability matrix. So either $v = 0$ (impossible since we assumed $v \neq 0$), or Q does not have full column rank (impossible, since (C, A) is observable). Therefore, we conclude that $\left\| C e^{At} v \right\|$ is not zero for all t and so the integral is strictly positive, i.e. $v^\mathsf{T} W v$.

3. Discrete-time Lyapunov equation. If all eigenvalues of A satisfy $|\lambda| < 1$, we say that A is Schur-stable or discrete-time stable. Prove the discrete-time counterpart to the Lyapunov theorem:

For any $Q = Q^\mathsf{T} \succ 0$, the equation $A^\mathsf{T} P A - P + Q = 0$ has a solution $P^\mathsf{T} = P \succ 0$ if and only if A is Schur-stable. Again, be sure to prove both directions.

Hint: To prove the *if* part, the P will involve an infinite sum rather than an infinite integral.

SOLUTION: ($\Longrightarrow$) Suppose $P = P^\mathsf{T} \succ 0$ and $A^\mathsf{T} P A - P + Q = 0$. Let (λ, x) be an eigenpair of A. Multiplying both sides of the equation by x^* and x:

$$\begin{aligned} & x^* \left(A^\mathsf{T} P A - P + Q \right) x = 0 \\ \Longrightarrow \quad & (\lambda \bar{\lambda} - 1) x^* P x + x^* Q x = 0 \\ \Longleftrightarrow \quad & \left(|\lambda|^2 - 1 \right) (x^* P x) + (x^* Q x) = 0 \end{aligned}$$

Since $P \succ 0$ and $Q \succ 0$, we have $x^* P x > 0$ and $x^* Q x > 0$. Therefore, we conclude that $(|\lambda|^2 - 1) < 0$, which implies that $|\lambda| < 1$.

($\Longleftarrow$) Suppose A is Schur-stable. Then $A^k \to 0$ as $k \to \infty$. Define P using the infinite sum:

$$P = \sum_{k=0}^\infty (A^k)^\mathsf{T} Q (A^k)$$

It's clear that $P = P^\mathsf{T}$. To show that $P \succ 0$, multiply both sides by some nonzero $v \in \mathbb{R}^n$.

$$v^\mathsf{T} P v = \sum_{k=0}^\infty (A^k v)^\mathsf{T} Q (A^k v)$$

Since $Q \succ 0$, each term of the sum is nonnegative. Moreover, the first term in the sum ($k = 0$) is simply $v^\mathsf{T} Q v$, which is strictly positive unless $v = 0$. We conclude that $P \succ 0$. We can also check that this P satisfies the matrix equation:

$$A^\mathsf{T} P A = \sum_{k=0}^\infty A^\mathsf{T} (A^k)^\mathsf{T} Q (A^k) A = \sum_{k=0}^\infty (A^{k+1})^\mathsf{T} Q (A^{k+1}) = P - Q$$

and this completes the proof.

4. Transient behavior. An asymptotically stable linear dynamical system with state $x(t)$ satisfies $\lim_{t \to \infty} \|x(t)\| = 0$. However, this convergence may not be monotonic. In other words, $\|x(t)\|$ can get very large before it settles down to zero.

a) As an example, consider the stable linear system $\dot{x} = \begin{bmatrix} -0.1 & 100 \\ 0 & -0.2 \end{bmatrix} x$ with $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Plot $\|x(t)\|$ as a function of t for $0 \le t \le 100$.

b) Suppose $\dot{x} = Ax$ is an asymptotically stable linear system. Let $P \succ 0$ be the solution to the Lyapunov equation $A^\mathsf{T} P + PA + I = 0$. Prove that if we define $z(t) = P^{1/2} x(t)$, then the transformed state $z(t)$ has the property that $\|z(t)\|$ converges monotonically to zero.

Note: $P^{1/2}$ is the matrix square root. It is defined as the unique symmetric positive definite matrix such that $P^{1/2} P^{1/2} = P$.

c) We will numerically verify the result of part b on the example system of part a. To do this, make use MATLAB's `lyap` function to solve the Lyapunov equation and `sqrtm` to find the matrix square root. Then, plot $\|z(t)\|$ as a function of t and verify that it's a decreasing function.

Note: Be aware that MATLAB uses a different convention (transposes!) than what we covered in class. Type `help lyap` to learn more about the syntax.

SOLUTION:

a) (see part c)

b) Note that $\|z\|^2 = \left\|P^{1/2} x\right\|^2 = x^\mathsf{T} P x$. Therefore, we can compute the rate of change of this quantity over time:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \|z(t)\|^2 &= \frac{\mathrm{d}}{\mathrm{d}t} x(t)^\mathsf{T} P x(t) \\
&= \dot{x}(t)^\mathsf{T} P x(t) + x(t)^\mathsf{T} P \dot{x}(t) \\
&= x(t)^\mathsf{T} \left(A^\mathsf{T} P + PA \right) x(t) \\
&= -\|x(t)\|^2 \le 0
\end{aligned}$$

Where in the last step we used the fact that $A^\mathsf{T} P + PA = -I$. Therefore $\|z(t)\|$ is decreasing as long as $x(t) \neq 0$.

c) Code and plot is given on the following page.

```
A = [-0.1 100; 0 -0.2];      % define system matrix
P = lyap(A',eye(2));         % solve Lyapunov equation

N = 1000;
t = linspace(0,100,N);    % generate time points

% initialize state
x0 = [0;1];
x = zeros(2,N);
xnorm = zeros(1,N);
znorm = zeros(1,N);

for k = 1:N
    x(:,k) = expm(A*t(k))*x0;
    xnorm(k) = norm(x(:,k));
end

% transform all the timepoints to z-coordinates
% and compute the resulting norm
z = sqrtm(P)*x;
for k = 1:N
    znorm(k) = norm(z(:,k));
end

figure(1);
subplot(211); plot(t,xnorm);
ylabel('$\|x(t)\|$','interpreter','latex')
subplot(212); plot(t,znorm)
xlabel('$t$','interpreter','latex');
ylabel('$\|z(t)\|$','interpreter','latex')
```

and here is the resulting plot. As expected, $\|x(t)\|$ shoots up before settling down while $\|z(t)\|$ is monotonically decreasing.