ECE 717
Homework 4: Stability

due: Wednesday October 30, 2019

1. **Warm-up.** Consider the linear system $\dot{x} = Ax$. For each case, examine the eigenvalues to determine whether the system is (1) asymptotically stable, (2) marginally stable, or (3) unstable.

   a) $A = \begin{bmatrix} 0 & 1 \\ -14 & -4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 0 & 1 \\ -14 & 4 \end{bmatrix}$ **c)** $A = \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix}$ **d)** $A = \begin{bmatrix} 0 & 1 \\ -14 & 0 \end{bmatrix}$

2. **A more general Lyapunov result.** The Lyapunov stability result states that for any $Q = Q^\mathsf{T} \succ 0$, the equation $A^\mathsf{T} P + PA + Q = 0$ has a solution $P = P^\mathsf{T} \succ 0$ if and only if A is Hurwitz (all eigenvalues of A have negative real part). Using a similar proof technique, show the following result:

   Suppose (C, A) is observable. Prove that the equation $A^\mathsf{T} W + WA + C^\mathsf{T} C = 0$ has a solution $W = W^\mathsf{T} \succ 0$ if and only if A is Hurwitz.

   Note: this result is more general than the one we saw in class, because typically we will have $C^\mathsf{T} C \succeq 0$ but not $C^\mathsf{T} C \succ 0$. Be sure to prove both directions (if *and* only if).

3. **Discrete-time Lyapunov equation.** If all eigenvalues of A satisfy $|\lambda| < 1$, we say that A is Schur-stable or discrete-time stable. Prove the discrete-time counterpart to the Lyapunov theorem:

   For any $Q = Q^\mathsf{T} \succ 0$, the equation $A^\mathsf{T} PA - P + Q = 0$ has a solution $P^\mathsf{T} = P \succ 0$ if and only if A is Schur-stable. Again, be sure to prove both directions.

   Hint: To prove the *if* part, the P will involve an infinite sum rather than an infinite integral.

4. **Transient behavior.** An asymptotically stable linear dynamical system with state $x(t)$ satisfies $\lim_{t\to\infty} \|x(t)\| = 0$. However, this convergence may not be monotonic. In other words, $\|x(t)\|$ can get very large before it settles down to zero.

   a) As an example, consider the stable linear system $\dot{x} = \begin{bmatrix} -0.1 & 100 \\ 0 & -0.2 \end{bmatrix} x$ with $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
   Plot $\|x(t)\|$ as a function of t for $0 \le t \le 100$.

   b) Suppose $\dot{x} = Ax$ is an asymptotically stable linear system. Let $P \succ 0$ be the solution to the Lyapunov equation $A^\mathsf{T} P + PA + I = 0$. Prove that if we define $z(t) = P^{1/2} x(t)$, then the transformed state $z(t)$ has the property that $\|z(t)\|$ converges monotonically to zero.
   Note: $P^{1/2}$ is the matrix square root. It is defined as the unique symmetric positive definite matrix such that $P^{1/2} P^{1/2} = P$.

   c) We will numerically verify the result of part b on the example system of part a. To do this, make use MATLAB's `lyap` function to solve the Lyapunov equation and `sqrtm` to find the matrix square root. Then, plot $\|z(t)\|$ as a function of t and verify that it's a decreasing function.
   Note: Be aware that MATLAB uses a different convention (transposes!) than what we covered in class. Type `help lyap` to learn more about the syntax.