# ECE 717

## Homework 2: State space solutions

due: Wednesday October 2, 2019

1. **Discretizing continuous equation.** In practice it is difficult to work with continuous-time signals because of storage and computational considerations. In this problem, we will discretize continuous state-space equations in a way that preserves the continuous-time behavior.

   **a)** Consider the continuous-time state-space system: $\dot{x}(t) = Ax(t)$ with initial condition $x(0) = x_0$. We would like to find an exact discretization of this system. Specifically, we will sample $x(t)$ at times $t = 0, h, 2h, \ldots$. To this effect, define the discrete-time signal

   $$x_d[k] = x(kh) \quad \text{for } k = 0, 1, \ldots$$

   Find a matrix $A_d$ such that $x_d[k+1] = A_d x_d[k]$. The matrix $A_d$ should depend on $A$ and $h$.

   **b)** Let's add an input signal. Suppose we have the discrete input $u_d[k]$, and define the continuous input signal a being constant between sampling times. In other words:

   $$u(t) = u_d[k] \qquad \text{for } kh \le t < (k+1)h$$

   Now consider the continuous-time system: $\dot{x}(t) = Ax(t) + Bu(t)$ using the piecewise-constant input defined above. Find matrices $A_d$ and $B_d$ such that $x_d[k+1] = A_d x_d[k] + B_d u_d[k]$. Here, $x_d[k]$ is defined the same as in part (a). You may express $B_d$ as an integral.

   **c)** Use MATLAB to simulate $x(t)$ for part (a) for the case where

   $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -9 & -5 \end{bmatrix}$$

   Plot all three components of $x(t)$ for $0 \le t \le 5$, with $x_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\mathsf{T}$. Then, simulate the discretized system using $h = 0.2$. Verify that the two simulations agree (the second one should be a sampled version of the first one). Note: the MATLAB command for computing matrix exponentials is `expm`.

**SOLUTION:**

**a)** The solution to the autonomous system is $x(t) = e^{At}x_0$. Substituting $t = kh$ and using the definition for $x_d[k]$, we obtain

$$x_d[k+1] = e^{A(k+1)h}x_0 = e^{Ah}e^{Akh}x_0 = e^{Ah}x_d[k]$$

So the discretized version has $A_d = e^{Ah}$.

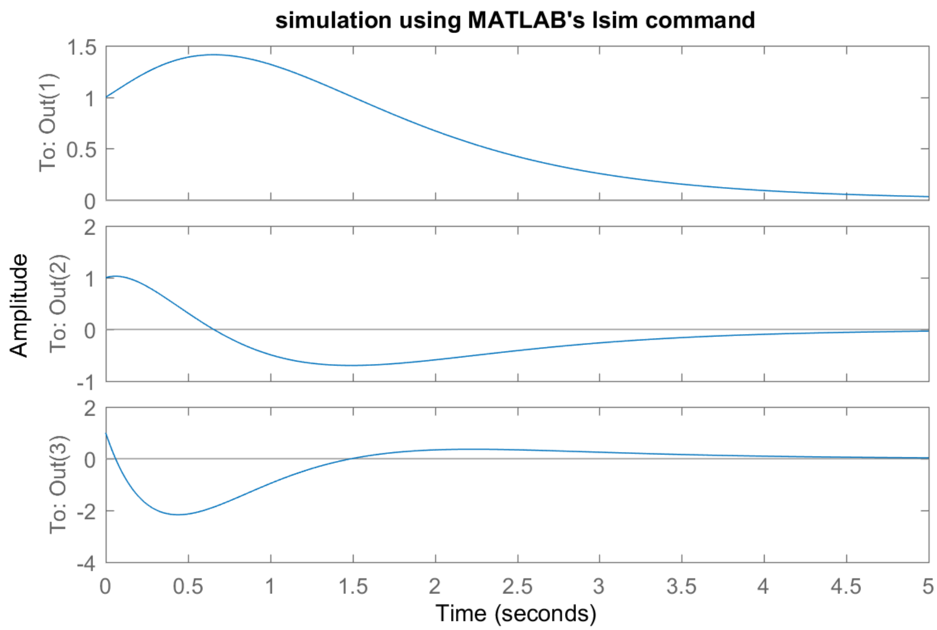
**b)** The solution to the autonomous system at time $t_2$ as a function of the initial time $t_1$ is given by $x(t_2) = e^{A(t_2-t_1)}x(t_1) + \int_{t_1}^{t_2} e^{A(t_2-\tau)}Bu(\tau)\,\mathrm{d}\tau$. Substituting $t_1 = kh$ and $t_2 = (k+1)h$ and using the definition for $x_d[k]$, we obtain

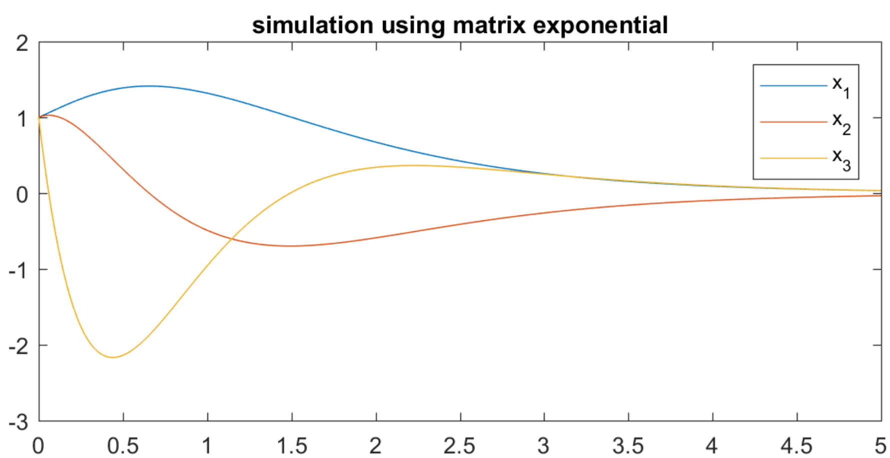
$$\begin{aligned} x_d[k+1] &= e^{Ah}x_d[k] + \int_{kh}^{kh+h} e^{A(kh+h-\tau)}Bu(\tau)\,\mathrm{d}\tau \\ &= e^{Ah}x_d[k] + \left(\int_{kh}^{kh+h} e^{A(kh+h-\tau)}B\,\mathrm{d}\tau\right)u_d[k] \\ &= e^{Ah}x_d[k] + \left(\int_0^h e^{A\tau}B\,\mathrm{d}\tau\right)u_d[k] \end{aligned}$$

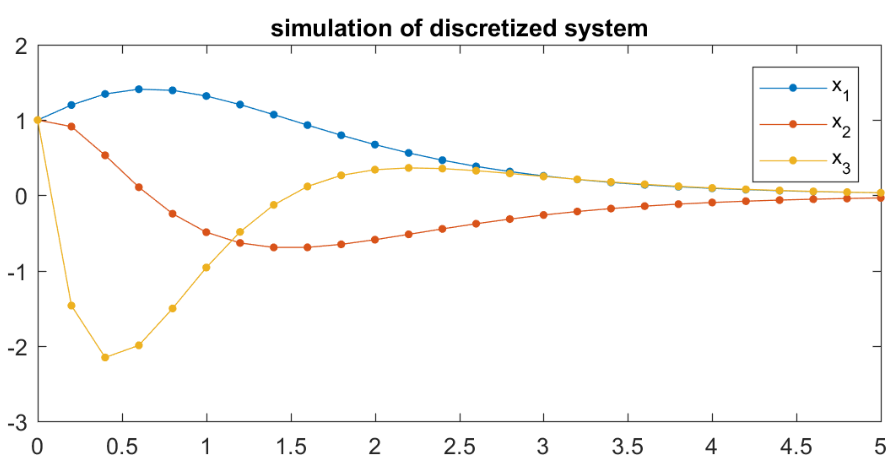
So the discretized version has $A_d = e^{Ah}$ and $B_d = \int_0^h e^{A\tau} B \, \mathrm{d}\tau$.

**c)** Here is code that simulates using MATLAB's built-in `lsim` function, then manually using the matrix exponential (continuous-time solution), and finally using the discretized system. We can see that the three agree.

```
A = [0 1 0; 0 0 1; -5 -9 -5];
x0 = [1; 1; 1];

% simulation using LSIM command. To do this, we make B zero (no input), we
% make C the identity (return the state as output).
sys = ss(A,zeros(3,1),eye(3),0);
t = linspace(0,5,300);
u = 0*t; % zero input
figure(1)
lsim(sys,u,t,x0)
title('simulation using MATLAB''s lsim command')

% let's repeat the simulation, but this time computing the solution
% ourselves using the matrix exponential.
t = linspace(0,5,300);
x = zeros(3,300);
for k = 1:300
   x(:,k) = expm(A*t(k))*x0;
end
figure(2)
plot(t,x)
title('simulation using matrix exponential')
legend('x_1','x_2','x_3')

% now let's discretize and see what we get:
h = 0.2;
Ad = expm(A*h);
td = 0:h:5;
n = numel(td);
xd = zeros(3,n);
xd(:,1) = x0;
for j = 1:n-1
   xd(:,j+1) = Ad*xd(:,j);
end
figure(3)
plot(td,xd,'.-','MarkerSize',12)
title('simulation of discretized system')
legend('x_1','x_2','x_3')
```


simulation using MATLAB's lsim command
Amplitude
To: Out(1)
To: Out(2)
To: Out(3)
Time (seconds)


simulation using matrix exponential
$x_1$
$x_2$
$x_3$


simulation of discretized system
$x_1$
$x_2$
$x_3$

**2. Dual state-space system.** Consider the MIMO system:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Suppose the transfer function of this system (from $u$ to $y$) is given by $H(s)$. Now consider the *dual* system with state $z(t)$ and state-space equations:

$$\dot{z}(t) = -A^{\mathsf{T}} z(t) - C^{\mathsf{T}} v(t)$$
$$w(t) = B^{\mathsf{T}} z(t) + D^{\mathsf{T}} v(t)$$

Prove that the transfer function for this new system (from $v$ to $w$) is given by $\big(H(-s)\big)^{\mathsf{T}}$.

**SOLUTION:** The transfer function for $(A, B, C, D)$ is $H(s) = C(sI - A)^{-1}B + D$. So the transfer function for the dual system $(-A^{\mathsf{T}}, -C^{\mathsf{T}}, B^{\mathsf{T}}, D^{\mathsf{T}})$ is: $B^{\mathsf{T}}(sI + A^{\mathsf{T}})^{-1}(-C^{\mathsf{T}}) + D^{\mathsf{T}}$. We can now check:

$$\begin{aligned}(H(-s))^{\mathsf{T}} &= \big(C(-sI - A)^{-1}B + D\big)^{\mathsf{T}} \\ &= B^{\mathsf{T}}(-sI - A)^{-\mathsf{T}}C^{\mathsf{T}} + D^{\mathsf{T}} \\ &= B^{\mathsf{T}}(sI + A^{\mathsf{T}})^{-1}(-C^{\mathsf{T}}) + D^{\mathsf{T}}\end{aligned}$$

and this is the same as the previous expression.

**3. Diagonal form.** Consider the state-space equation $\dot{x}(t) = Ax(t) + Bu(t)$ where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

**a)** Use a state transformation to convert this system into diagonal canonical form.

**b)** Is this system controllable? Explain your answer by analyzing your solution to part (a).

**c)** Compute closed-form expressions for $A^k$ and for $e^{At}$.

**SOLUTION:**

**a)** Based on the form of the $A$ matrix, we know the characteristic polynomial is

$$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = (\lambda + 1)(\lambda + 2)(\lambda + 3)$$

So the eigenvalues of $A$ are $\{-1, -2, -3\}$. One possible diagonalization is:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Computing the inverse, we have:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 6 & 5 & 1 \\ -6 & -8 & -2 \\ 2 & 3 & 1 \end{bmatrix}$$

The diagonal form of the system is: $(T^{-1}AT, T^{-1}B)$, which we can compute to be:

$$\widehat{A} = T^{-1}AT = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \qquad \text{and} \qquad \widehat{B} = T^{-1}B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

**b)** The system is not controllable. The controllability matrix (which we may compute in the transformed coordinates) is:

$$\begin{bmatrix} \widehat{B} & \widehat{A}\widehat{B} & \widehat{A}^2\widehat{B} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

and this matrix is clearly rank-deficient because it has a row of zeros.

**c)** We can also compute powers and exponentials using the diagonalized form:

$$A^k = \frac{1}{2} \begin{bmatrix} 6 & 5 & 1 \\ -6 & -8 & -2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} (-1)^k & 0 & 0 \\ 0 & (-2)^k & 0 \\ 0 & 0 & (-3)^k \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$e^{At} = \frac{1}{2} \begin{bmatrix} 6 & 5 & 1 \\ -6 & -8 & -2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

4. **Linear time-varying problem.** Consider the time-*varying* linear dynamical system:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \qquad \text{with } x(0) = x_0 \tag{1}$$

**a)** Suppose we can find a matrix function $P(t)$ with the property that

$$\dot{P}(t) = A(t)P(t) \qquad \text{and } \det(P(t)) \neq 0 \quad \text{for all } t$$

A $P(t)$ with this property is called a *fundamental matrix*. Note that in the case where $A(t)$ is constant, we can use $P(t) = e^{At}$, as seen in class. Define $\Phi(t, \tau) = P(t)P(\tau)^{-1}$. Prove that the solution to (1) is given by:

$$y(t) = C(t)\Phi(t, 0)x_0 + \int_0^t C(t)\Phi(t, \tau)B(\tau)u(\tau)\,\mathrm{d}\tau + D(t)u(t)$$

**Hint:** Use the transformation $x(t) = P(t)z(t)$.

**b)** Based on the result from part (a), it's enough to find a fundamental matrix $P(t)$ and we automatically obtain a solution to the time-varying system (1). Consider the special case where $x(t)$ is a scalar. In this case, $A(t) = a(t)$ is also a scalar, and the fundamental matrix $P(t) = p(t)$ is a scalar as well. Find $p(t)$.

**c)** Consider the sequence of functions $\{M_k\}$ defined by:

$$\begin{aligned} M_0(t) &= I \\ M_{k+1}(t) &= I + \int_0^t A(\tau)M_k(\tau)\,\mathrm{d}\tau \qquad \text{for } k = 0, 1, \ldots \end{aligned}$$

and let $P(t) = \lim_{k\to\infty} M_k(t)$. In other words,

$$\begin{aligned} P(t) = I \;+ \int_0^t A(\tau_1)\,\mathrm{d}\tau_1 \;+ \int_0^t A(\tau_1)\int_0^{\tau_1} A(\tau_2)\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1 \\ + \int_0^t A(\tau_1)\int_0^{\tau_1} A(\tau_2)\int_0^{\tau_2} A(\tau_3)\,\mathrm{d}\tau_3\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1 \;+\; \cdots \end{aligned}$$

It turns out this is a well-defined limit (the sequence $\{M_k\}$ converges uniformly). Prove that $P(t)$ is a fundamental matrix.

**SOLUTION:**

**a)** Using the transformation $x(t) = P(t)z(t)$, we have:

$$\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(P(t)z(t)\Big) = \dot{P}(t)z(t) + P(t)\dot{z}(t) = A(t)P(t)z(t) + P(t)\dot{z}(t)$$

Substituting into the equation (1), we obtain the equations:

$$\begin{aligned} A(t)P(t)z(t) + P(t)\dot{z}(t) &= A(t)P(t)z(t) + B(t)u(t) \\ y(t) &= C(t)P(t)z(t) + D(t)u(t) \end{aligned}$$

The first equation has cancellations and simplifies to $\dot{z}(t) = P(t)^{-1}B(t)u(t)$. Therefore, we can integrate and conclude that:

$$z(t) = \int_0^t P(\tau)^{-1}B(\tau)u(\tau)\,\mathrm{d}\tau$$

Substituting this into the second equation and letting $\Phi(t, \tau) = P(t)P(\tau)^{-1}$ gives us the required result.

**b)** If $A(t) = a(t)$ is a scalar, the fundamental matrix is also a scalar $P(t) = p(t)$. We seek to solve the equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} p(t) = a(t)p(t)$$

We saw in class how to solve this using an integrating factor. Another way is by writing it in differential form and integrating:

$$\begin{aligned}\frac{\mathrm{d}p}{p} = a(t)\,\mathrm{d}t &\implies \int_{p(0)}^{p(t)} \frac{1}{p}\,\mathrm{d}p = \int_0^t a(\tau)\,\mathrm{d}\tau \\ &\implies \log(p(t)) - \log(p(0)) = \int_0^t a(\tau)\,\mathrm{d}\tau \\ &\implies p(t) = p(0)e^{\int_0^t a(\tau)\,\mathrm{d}\tau}\end{aligned}$$

We can use any $p(0) \neq 0$. Ultimately, it doesn't matter since it will cancel out when we compute

$$\Phi(t_1, t_2) = \frac{p(t_1)}{p(t_2)} = e^{\int_{t_2}^{t_1} a(\tau)\,\mathrm{d}\tau}$$

**c)** The function $P(t)$ is a series: $P(t) = \sum_{k=0}^{\infty} P_k(t)$. The terms in this series are:

$$\begin{aligned}P_0(t) &= I \\ P_1(t) &= \int_0^t A(\tau_1)\,\mathrm{d}\tau \\ P_k(t) &= \int_0^t A(\tau_1) \int_0^{\tau_1} A(\tau_2) \int_0^{\tau_2} A(\tau_3) \cdots \int_0^{\tau_k} A(\tau_k)\,\mathrm{d}\tau_k \cdots \mathrm{d}\tau_3\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1\end{aligned}$$

Differentiating a general term with respect to $t$, we obtain $\frac{\mathrm{d}}{\mathrm{d}t}P_0(t) = 0$ and for $k \geq 1$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} P_k(t) &= A(t) \int_0^t A(\tau_2) \int_0^{\tau_2} A(\tau_3) \cdots \int_0^{\tau_{k-1}} A(\tau_k)\,\mathrm{d}\tau_k \cdots \mathrm{d}\tau_3\,\mathrm{d}\tau_2 \\ &= A(t) \int_0^t A(\tau_1) \int_0^{\tau_1} A(\tau_2) \cdots \int_0^{\tau_{k-2}} A(\tau_{k-1})\,\mathrm{d}\tau_{k-1} \cdots \mathrm{d}\tau_2\,\mathrm{d}\tau_1 \\ &= A(t)P_{k-1}(t)\end{aligned}$$

Where in the second step, we relabeled the dummy integration variables. Substituting this in to the series definition, we get:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} P(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \sum_{k=0}^{\infty} P_k(t) \\ &= \sum_{k=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t} P_k(t) \\ &= \sum_{k=1}^{\infty} A(t)P_{k-1}(t) \\ &= A(t) \sum_{k=0}^{\infty} P_k(t) \\ &= A(t)P(t)\end{aligned}$$