ECE 717
Homework 2: State space solutions

due: Wednesday October 2, 2019

1. **Discretizing continuous equation.** In practice it is difficult to work with continuous-time signals because of storage and computational considerations. In this problem, we will discretize continuous state-space equations in a way that preserves the continuous-time behavior.

   a) Consider the continuous-time state-space system: $\dot{x}(t) = Ax(t)$ with initial condition $x(0) = x_0$. We would like to find an exact discretization of this system. Specifically, we will sample $x(t)$ at times $t = 0, h, 2h, \ldots$. To this effect, define the discrete-time signal

   $$x_d[k] = x(kh) \quad \text{for } k = 0, 1, \ldots$$

   Find a matrix A_d such that $x_d[k+1] = A_d x_d[k]$. The matrix A_d should depend on A and h.

   b) Let's add an input signal. Suppose we have the discrete input $u_d[k]$, and define the continuous input signal a being constant between sampling times. In other words:

   $$u(t) = u_d[k] \qquad \text{for } kh \le t < (k+1)h$$

   Now consider the continuous-time system: $\dot{x}(t) = Ax(t) + Bu(t)$ using the piecewise-constant input defined above. Find matrices A_d and B_d such that $x_d[k+1] = A_d x_d[k] + B_d u_d[k]$. Here, $x_d[k]$ is defined the same as in part (a). You may express B_d as an integral.

   c) Use MATLAB to simulate $x(t)$ for part (a) for the case where

   $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -9 & -5 \end{bmatrix}$$

   Plot all three components of $x(t)$ for $0 \le t \le 5$, with $x_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\mathsf{T}$. Then, simulate the discretized system using $h = 0.2$. Verify that the two simulations agree (the second one should be a sampled version of the first one). Note: the MATLAB command for computing matrix exponentials is `expm`.

2. **Dual state-space system.** Consider the MIMO system:

   $$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

   Suppose the transfer function of this system (from u to y) is given by $H(s)$. Now consider the *dual* system with state $z(t)$ and state-space equations:

   $$\begin{aligned} \dot{z}(t) &= -A^\mathsf{T} z(t) - C^\mathsf{T} v(t) \\ w(t) &= B^\mathsf{T} z(t) + D^\mathsf{T} v(t) \end{aligned}$$

   Prove that the transfer function for this new system (from v to w) is given by $\big(H(-s)\big)^\mathsf{T}$.

3. Diagonal form. Consider the state-space equation $\dot{x}(t) = Ax(t) + Bu(t)$ where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

a) Use a state transformation to convert this system into diagonal canonical form.

b) Is this system controllable? Explain your answer by analyzing your solution to part (a).

c) Compute closed-form expressions for A^k and for e^{At}.

4. Linear time-varying problem. Consider the time-*varying* linear dynamical system:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \qquad \text{with } x(0) = x_0 \tag{1}$$

a) Suppose we can find a matrix function $P(t)$ with the property that

$$\dot{P}(t) = A(t)P(t) \qquad \text{and } \det(P(t)) \neq 0 \quad \text{for all } t$$

A $P(t)$ with this property is called a *fundamental matrix*. Note that in the case where $A(t)$ is constant, we can use $P(t) = e^{At}$, as seen in class. Define $\Phi(t,\tau) = P(t)P(\tau)^{-1}$. Prove that the solution to (1) is given by:

$$y(t) = C(t)\Phi(t,0)x_0 + \int_0^t C(t)\Phi(t,\tau)B(\tau)u(\tau)\,\mathrm{d}\tau + D(t)u(t)$$

Hint: Use the transformation $x(t) = P(t)z(t)$.

b) Based on the result from part (a), it's enough to find a fundamental matrix $P(t)$ and we automatically obtain a solution to the time-varying system (1). Consider the special case where $x(t)$ is a scalar. In this case, $A(t) = a(t)$ is also a scalar, and the fundamental matrix $P(t) = p(t)$ is a scalar as well. Find $p(t)$.

c) Consider the sequence of functions $\{M_k\}$ defined by:

$$\begin{aligned} M_0(t) &= I \\ M_{k+1}(t) &= I + \int_0^t A(\tau)M_k(\tau)\,\mathrm{d}\tau \qquad \text{for } k = 0, 1, \ldots \end{aligned}$$

and let $P(t) = \lim_{k\to\infty} M_k(t)$. In other words,

$$P(t) = I + \int_0^t A(\tau_1)\,\mathrm{d}\tau_1 + \int_0^t A(\tau_1)\int_0^{\tau_1} A(\tau_2)\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1 + \int_0^t A(\tau_1)\int_0^{\tau_1} A(\tau_2)\int_0^{\tau_2} A(\tau_3)\,\mathrm{d}\tau_3\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1 + \cdots$$

It turns out this is a well-defined limit (the sequence $\{M_k\}$ converges uniformly). Prove that $P(t)$ is a fundamental matrix.