

## ECE 717

### Homework 1: State space models

due: Wednesday September 18, 2019

1. **State-space models.** Find state-space realizations for each of the following linear systems.

a) The transfer function:

$$\frac{Y(s)}{U(s)} = \frac{s^3 + s - 1}{3s^3 + 2s^2 - s + 2}$$

b) The system of differential equations:

$$\begin{aligned} \ddot{y}_1(t) + 5\dot{y}_1(t) - 10(y_2(t) - y_1(t)) &= u_1(t) \\ 2\ddot{y}_2(t) + \dot{y}_2(t) + 10(y_2(t) - y_1(t)) &= u_2(t) \end{aligned}$$

Note: there are two inputs  $(u_1, u_2)$  and two outputs  $(y_1, y_2)$  in this case.

c) The Fibonacci sequence, which is an autonomous discrete-time system defined by the recurrence

$$F_n = F_{n-1} + F_{n-2}$$

#### SOLUTION:

a) Normalizing so we have  $a_n = 1$ , and performing polynomial division, we obtain:

$$\frac{s^3 + s - 1}{3s^3 + 2s^2 - s + 2} = \frac{\frac{1}{3}s^3 + \frac{1}{3}s - \frac{1}{3}}{s^3 + \frac{2}{3}s^2 - \frac{1}{3}s + \frac{2}{3}} = \frac{1}{3} + \frac{-\frac{2}{9}s^2 + \frac{4}{9}s - \frac{5}{9}}{s^3 + \frac{2}{3}s^2 - \frac{1}{3}s + \frac{2}{3}}$$

From here, we can read off the coefficients, and we obtain the state space realization:

$$\left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[ \begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & 1 \\ -\frac{5}{9} & \frac{4}{9} & -\frac{2}{9} & \frac{1}{3} \end{array} \right]$$

b) Define the state to be  $x = [y_1 \ y_2 \ \dot{y}_1 \ \dot{y}_2]^\top$ . Then we can write the equations as:

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -5x_1 + 10(x_2 - x_1) + u_1 \\ \dot{x}_4 &= \frac{1}{2}(-x_4 - 10(x_2 - x_1) + u_2) \end{aligned}$$

In state-space form, this becomes:

$$\left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[ \begin{array}{cccc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -15 & 10 & 0 & 0 & 1 & 0 \\ 5 & -5 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right]$$

c) Define the state to be  $x[k] = [F_{k-1} \quad F_k]^\top$ . The equations are therefore:

$$\begin{aligned}x_1[k+1] &= x_2[k] \\x_2[k+1] &= x_1[k] + x_2[k]\end{aligned}$$

So the (discrete-time) state-space equations are simply:

$$x[k+1] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x[k]$$

**2. Laplace and Z-transform.** These are standard results about Laplace and Z-transforms, but it's interesting to see them side-by-side and note the similarities!

a) For a function  $y : \mathbb{R} \rightarrow \mathbb{R}$ , the Laplace transform is defined as:

$$\mathcal{L}\{y\}(s) = Y(s) = \int_0^\infty e^{-s\tau} y(\tau) d\tau$$

Suppose that  $y$  is differentiable. Prove that  $\mathcal{L}\{\dot{y}\}(s) = sY(s) - y(0)$  when  $Y(s)$  is well-defined. Here, "well-defined" just means that  $s$  is such that the integral converges.

b) For a function  $x : \mathbb{Z} \rightarrow \mathbb{R}$ , the Z-transform is defined as:

$$\mathcal{Z}\{x\}(z) = X(z) = \sum_{k=0}^{\infty} x[k]z^{-k}$$

Define  $x_+$  as the shifted sequence:  $x_+[k] = x[k+1]$ . Prove that  $\mathcal{Z}\{x_+\}(z) = zX(z) - zx[0]$  when  $X(z)$  is well-defined. Here, "well-defined" means that  $z$  is such that the sum converges.

**SOLUTION:**

a) used integration by parts in the first step and assume that  $\lim_{\tau \rightarrow \infty} e^{-s\tau} y(\tau) = 0$ :

$$\begin{aligned}\mathcal{L}\{\dot{y}\}(s) &= \int_0^\infty e^{-s\tau} \dot{y}(\tau) d\tau = \left[ e^{-s\tau} y(\tau) \right]_{\tau=0}^{\tau=\infty} + \int_0^\infty s e^{-s\tau} y(\tau) d\tau \\ &= -y(0) + s \int_0^\infty e^{-s\tau} y(\tau) d\tau \\ &= sY(s) - y(0)\end{aligned}$$

b) Perform some manipulations to the definition and obtain:

$$\begin{aligned}\mathcal{Z}\{x_+\}(z) &= \sum_{k=0}^{\infty} x_+[k]z^{-k} = \sum_{k=0}^{\infty} x[k+1]z^{-k} \\ &= \sum_{k=1}^{\infty} x[k]z^{-(k-1)} \\ &= z \left( -x[0] + \sum_{k=0}^{\infty} x[k]z^{-k} \right) \\ &= zX(z) - zx[0]\end{aligned}$$

**3. Rigid-body dynamics.** When a rigid cylinder is freely rotating in space, it is subject to the Euler equations of motion. If we fix a coordinate frame to the cylinder's center of mass with the  $z$ -axis aligned with the axis of rotational symmetry, the equations of motions in three dimensions are:

$$\begin{aligned} I_p \dot{x}_1(t) &= (I_p - I_q)x_2(t)x_3(t) + u_1(t) \\ I_p \dot{x}_2(t) &= (I_q - I_p)x_1(t)x_3(t) + u_2(t) \\ I_q \dot{x}_3(t) &= u_3(t) \end{aligned}$$

Here,  $(x_1(t), x_2(t), x_3(t))$  and  $(u_1(t), u_2(t), u_3(t))$  are the angular velocities and applied torques in the fixed coordinate frame, respectively. You can think of the applied torques as inputs (e.g. from gyroscopes) and the angular velocities as state variables. The constants  $I_p > 0$  and  $I_q > 0$  are the moments of inertia of the cylinder and you may assume they are known constants.

- a) Consider the nominal state values  $\tilde{x}_1 = \tilde{x}_2 = 0$ ,  $\tilde{x}_3 = \omega_0$  for some fixed  $\omega_0 > 0$  and nominal input values  $\tilde{u}_1 = \tilde{u}_2 = \tilde{u}_3 = 0$ . Verify that  $\tilde{x}$  and  $\tilde{u}$  satisfy the equations of motion. Find linearized state-space equations about the nominal values  $(\tilde{x}, \tilde{u})$ .
- b) We will now consider a *time-varying* nominal trajectory. Show that the trajectory:

$$\tilde{x}_1(t) = \sin\left(\left(1 - \frac{I_q}{I_p}\right)\omega_0 t\right), \quad \tilde{x}_2(t) = \cos\left(\left(1 - \frac{I_q}{I_p}\right)\omega_0 t\right), \quad \tilde{x}_3(t) = \omega_0$$

satisfies the equations of motion when  $\tilde{u}_1(t) = \tilde{u}_2(t) = \tilde{u}_3(t) = 0$  for all  $t$  (no input). As in part a,  $\omega_0 > 0$  is a fixed constant.

- c) Linearize the equations of motion about the time-varying nominal trajectory  $\tilde{x}(t)$  from part b. Express your answer as state-space equations where the inputs are  $(\delta u_1(t), \delta u_2(t), \delta u_3(t))$  and the states are the perturbations of angular momentum from the nominal trajectory  $(\delta x_1(t), \delta x_2(t), \delta x_3(t))$ . Hint: the solution will be a linear *time-varying* system.

### SOLUTION:

- a) When substituting, the time-derivatives are zero because  $\tilde{x}$  is a constant. The only nonzero variable is  $\tilde{x}_3$ , but it gets multiplied by  $\tilde{x}_1$  or  $\tilde{x}_2$ , both of which are zero. The first equation is of the form:

$$I_p \dot{x}_1 = f(x_1, x_2, x_3, u_1, u_2, u_3)$$

Linearizing yields:

$$\begin{aligned} I_p \delta \dot{x}_1 &= \left[ \frac{\partial f}{\partial x_1}(\tilde{x}, \tilde{u}) \right] \delta x_1 + \left[ \frac{\partial f}{\partial x_2}(\tilde{x}, \tilde{u}) \right] \delta x_2 + \left[ \frac{\partial f}{\partial x_3}(\tilde{x}, \tilde{u}) \right] \delta x_3 \\ &\quad + \left[ \frac{\partial f}{\partial u_1}(\tilde{x}, \tilde{u}) \right] \delta u_1 + \left[ \frac{\partial f}{\partial u_2}(\tilde{x}, \tilde{u}) \right] \delta u_2 + \left[ \frac{\partial f}{\partial u_3}(\tilde{x}, \tilde{u}) \right] \delta u_3 \\ &= [0] \delta x_1 + [(I_p - I_q)\omega_0] \delta x_2 + [0] \delta x_3 + [1] \delta u_1 + [0] \delta u_2 + [0] \delta u_3 \end{aligned}$$

Doing something similar for the other equations, we obtain:

$$\begin{bmatrix} I_p \delta \dot{x}_1 \\ I_p \delta \dot{x}_2 \\ I_q \delta \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & (I_p - I_q)\omega_0 & 0 \\ (I_q - I_p)\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{bmatrix}$$

Dividing through by  $I_p$  and  $I_q$  (multiplying by the inverse), we obtain:

$$\begin{bmatrix} \delta \dot{x}_1 \\ \delta \dot{x}_2 \\ \delta \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & I_p^{-1}(I_p - I_q)\omega_0 & 0 \\ I_p^{-1}(I_q - I_p)\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix} + \begin{bmatrix} I_p^{-1} & 0 & 0 \\ 0 & I_p^{-1} & 0 \\ 0 & 0 & I_q^{-1} \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{bmatrix}$$

b) Substituting the nominal trajectory into the first equation, we obtain:

$$I_p \dot{\tilde{x}}_1(t) = I_p \left(1 - \frac{I_q}{I_p}\right) \omega_0 \cos \left( \left(1 - \frac{I_q}{I_p}\right) \omega_0 t \right) = (I_p - I_q) \tilde{x}_2(t) \tilde{x}_3(t) + \tilde{u}_1(t)$$

and we can do something similar for the other two equations.

c) The method is exactly analogous to part a, except this time when we substitute  $\tilde{x}$  and  $\tilde{u}$ , these are functions of time. The result is:

$$\begin{bmatrix} I_p \delta \dot{x}_1 \\ I_p \delta \dot{x}_2 \\ I_q \delta \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & (I_p - I_q) \omega_0 & (I_p - I_q) \cos \left( \left(1 - \frac{I_q}{I_p}\right) \omega_0 t \right) \\ (I_q - I_p) \omega_0 & (I_q - I_p) \sin \left( \left(1 - \frac{I_q}{I_p}\right) \omega_0 t \right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{bmatrix}$$

As in part a, we can divide through by  $I_p$  and  $I_q$  and we obtain a linear *time-varying* system of the form:

$$\delta \dot{x}(t) = A(t) \delta x(t) + B u(t)$$

**4. State-space simulation.** In this problem, we will use MATLAB to simulate a state-space system. Consider the spring-mass-damper model from lecture:

$$\ddot{y} + c\dot{y} + ky = u$$

- Let's pick values of  $c = 0.2$  and  $k = 1$ . Create a transfer function model for this system in MATLAB using the command `tf`. This can be done by specifying numerator and denominator coefficients or by using `s` directly. You can read the documentation and see examples by running `doc tf`. Plot the impulse response for  $0 \leq t \leq 60$  using the `impz` command, and repeat the task using  $c = 0.01$  (less damping) and  $c = 0.4$  (more damping). Describe what you see.
- This time, create a state-space model for this system in MATLAB using the command `ss`. Start by finding the  $(A, B, C, D)$  matrices by hand. Again, refer to `doc ss` for guidelines. Then, plot the impulse response as in part a and verify that you get the same results.
- Matlab can convert between state-space and transfer function for you. If `H` is the state-space object from part b, run `tf(H)` and verify that you recover the transfer function from part a. Likewise, if `G` is the transfer function object from part a, run `ss(G)` to find a state-space realization. Is this realization the same as `H`? Explain.

## SOLUTION:

Here is MATLAB code that generates the plots:

```

figure;
t_final = 60;
cvalues = [0.01, 0.2, 0.4]; % possible c values to test

for i = 1:3

    c = cvalues(i); % set value of c
    k = 1; % set value of k

    % Transfer function specification
    H1 = tf([1], [1 c k]) % First method: using num/den coefficients

    s = tf('s');
    H1 = 1/(s^2 + c*s + k) % Second method: direct specification

    % State-space specification
    A = [0 1; -k -c];
    B = [0; 1];
    C = [1 0];
    D = [0];
    H2 = ss(A,B,C,D) % state-space model

    subplot(3,1,i)
    impulse(H1,t_final)
    title(['Impulse response for k = ' num2str(k) ' and c = ' num2str(c)])
end

```

And here is the plot that was produced:

