ECE 717
Homework 1: State space models

due: Wednesday September 18, 2019

1. State-space models. Find state-space realizations for each of the following linear systems.

a) The transfer function:

$$\frac{Y(s)}{U(s)} = \frac{s^3 + s - 1}{3s^3 + 2s^2 - s + 2}$$

b) The system of differential equations:

$$\begin{aligned} \ddot{y}_1(t) + 5\dot{y}_1(t) - 10(y_2(t) - y_1(t)) &= u_1(t) \\ 2\ddot{y}_2(t) + \dot{y}_2(t) + 10(y_2(t) - y_1(t)) &= u_2(t) \end{aligned}$$

Note: there are two inputs (u_1, u_2) and two outputs (y_1, y_2) in this case.

c) The Fibonacci sequence, which is an autonomous discrete-time system defined by the recurrence

$$F_n = F_{n-1} + F_{n-2}$$

2. Laplace and Z-transform. These are standard results about Laplace and Z-transforms, but it's interesting to see them side-by-side and note the similarities!

a) For a function $y : \mathbb{R} \to \mathbb{R}$, the Laplace transform is defined as:

$$\mathcal{L}\{y\}(s) = Y(s) = \int_0^\infty e^{-s\tau} y(\tau)\, \mathrm{d}\tau$$

Suppose that y is differentiable. Prove that $\mathcal{L}\{\dot{y}\}(s) = sY(s) - y(0)$ when $Y(s)$ is well-defined. Here, "well-defined" just means that s is such that the integral converges.

b) For a function $x : \mathbb{Z} \to \mathbb{R}$, the Z-transform is defined as:

$$\mathcal{Z}\{x\}(z) = X(z) = \sum_{k=0}^{\infty} x[k] z^{-k}$$

Define x_+ as the shifted sequence: $x_+[k] = x[k+1]$. Prove that $\mathcal{Z}\{x_+\}(z) = zX(z) - zx[0]$ when $X(z)$ is well-defined. Here, "well-defined" means that z is such that the sum converges.

3. Rigid-body dynamics. When a rigid cylinder is freely rotating in space, it is subject to the Euler equations of motion. If we fix a coordinate frame to the cylinder's center of mass with the z-axis aligned with the axis of rotational symmetry, the equations of motions in three dimensions are:

$$\begin{aligned} I_p \dot{x}_1(t) &= (I_p - I_q) x_2(t) x_3(t) + u_1(t) \\ I_p \dot{x}_2(t) &= (I_q - I_p) x_1(t) x_3(t) + u_2(t) \\ I_q \dot{x}_3(t) &= u_3(t) \end{aligned}$$

Here, $(x_1(t), x_2(t), x_3(t))$ and $(u_1(t), u_2(t), u_3(t))$ are the angular velocities and applied torques in the fixed coordinate frame, respectively. You can think of the applied torques as inputs (e.g. from gyroscopes) and the angular velocities as state variables. The constants $I_p > 0$ and $I_q > 0$ are the moments of inertia of the cylinder and you may assume they are known constants.

a) Consider the nominal state values $\tilde{x}_1 = \tilde{x}_2 = 0$, $\tilde{x}_3 = \omega_0$ for some fixed $\omega_0 > 0$ and nominal input values $\tilde{u}_1 = \tilde{u}_2 = \tilde{u}_3 = 0$. Verify that $\tilde{x}$ and $\tilde{u}$ satisfy the equations of motion. Find linearized state-space equations about the nominal values $(\tilde{x}, \tilde{u})$.

b) We will now consider a *time-varying* nominal trajectory. Show that the trajectory:

$$\tilde{x}_1(t) = \sin\left(\left(1 - \tfrac{I_q}{I_p}\right)\omega_0 t\right), \qquad \tilde{x}_2(t) = \cos\left(\left(1 - \tfrac{I_q}{I_p}\right)\omega_0 t\right), \qquad \tilde{x}_3(t) = \omega_0$$

satisfies the equations of motion when $\tilde{u}_1(t) = \tilde{u}_2(t) = \tilde{u}_3(t) = 0$ for all t (no input). As in part a, $\omega_0 > 0$ is a fixed constant.

c) Linearize the equations of motion about the time-varying nominal trajectory $\tilde{x}(t)$ from part b. Express your answer as state-space equations where the inputs are $(\delta u_1(t), \delta u_2(t), \delta u_3(t))$ and the states are the perturbations of angular momentum from the nominal trajectory $(\delta x_1(t), \delta x_2(t), \delta x_3(t))$. Hint: the solution will be a linear time-*varying* system.

4. State-space simulation. In this problem, we will use MATLAB to simulate a state-space system. Consider the spring-mass-damper model from lecture:

$$\ddot{y} + c\dot{y} + ky = u$$

a) Let's pick values of $c = 0.2$ and $k = 1$. Create a transfer function model for this system in MATLAB using the command `tf`. This can be done by specifying numerator and denominator coefficients or by using `s` directly. You can read the documentation and see examples by running `doc tf`. Plot the impulse response for $0 \le t \le 60$ using the `impulse` command, and repeat the task using $c = 0.01$ (less damping) and $c = 0.4$ (more damping). Describe what you see.

b) This time, create a state-space model for this system in MATLAB using the command `ss`. Start by finding the (A, B, C, D) matrices by hand. Again, refer to `doc ss` for guidelines. Then, plot the impulse response as in part a and verify that you get the same results.

c) Matlab can convert between state-space and transfer function for you. If `H` is the state-space object from part b, run `tf(H)` and verify that you recover the transfer function from part a. Likewise, if `G` is the transfer function object from part a, run `ss(G)` to find a state-space realization. Is this realization the same as `H`? Explain.