ECE 717
Exam 3 – Fall 2019

SOLUTIONS

Name: ______________________________________

1. _____________

2. _____________

3. _____________

4. _____________

5. _____________

_____________ **(total score)**

1. Minimal realizations [10 points]. Consider the system with state-space realization:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}1 & 5\\ 8 & 4\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}-1\\ 2\end{bmatrix}u(t)$$
$$y(t) = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

a) Show that this realization is not minimal.

SOLUTION: The observability matrix is:

$$\begin{bmatrix}C\\ CA\end{bmatrix} = \begin{bmatrix}1 & 1\\ 9 & 9\end{bmatrix}$$

This matrix is not full-rank (rank 1) so the system is not observable and therefore not minimal.

b) Find a minimal realization for this system.

SOLUTION: Use the transformation:

$$T = \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

Now calculate:

$$T^{-1}AT = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 5\\ 8 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}-4 & 5\\ 4 & 4\end{bmatrix} = \begin{bmatrix}-4 & 1\\ 0 & 9\end{bmatrix}$$
$$T^{-1}B = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}-1\\ 2\end{bmatrix} = \begin{bmatrix}-1\\ 1\end{bmatrix}$$
$$CT = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\end{bmatrix}$$

So the transformed realization is:

$$\left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right] = \left[\begin{array}{cc|c} -4 & 1 & -1 \\ 0 & 9 & 1 \\ \hline 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{c|c} 9 & 1 \\ \hline 1 & 0 \end{array}\right]$$

(first state is unobservable so it can be removed).

ALT. SOLUTION: Compute the transfer function:

$$\begin{aligned} G(s) = C(sI-A)^{-1}B + D &= \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}s-1 & -5\\ -8 & s-4\end{bmatrix}^{-1}\begin{bmatrix}-1\\ 2\end{bmatrix} \\ &= \frac{1}{(s-1)(s-4)-40}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}s-4 & 5\\ 8 & s-1\end{bmatrix}\begin{bmatrix}-1\\ 2\end{bmatrix} \\ &= \frac{1}{s^2-5s-36}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}6-s\\ 2s-10\end{bmatrix} = \frac{(s-4)}{(s-4)(s+9)} = \frac{1}{s+9} \end{aligned}$$

where we canceled $(s-4)$ in the last step. So a minimal realization is $\left[\begin{array}{c|c} 9 & 1 \\ \hline 1 & 0 \end{array}\right]$.

2. Inverting an LTI system [10 points]. Consider the LTI system governed by the dynamics:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, and $D \in \mathbb{R}^{m \times m}$ is an invertible matrix. Note that the system is *square* (same number of inputs as outputs). Find a state-space realization for the *inverse* of this system. The *inverse* is an LTI system where the input and output are swapped (the input is y and the output is u) and the same dynamics as above are satisfied.

SOLUTION:

Rearrange the last equation to turn it into measurement equation for u:

$$u = -D^{-1}Cx + D^{-1}y$$

Now substitute this into the first equation to eliminate u:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ &= Ax + B(-D^{-1}Cx + D^{-1}y) \\ &= (A - BD^{-1}C)x + BD^{-1}y \end{aligned}$$

Putting both pieces together, we obtain a state-space realization for the inverse:

$$\begin{aligned} \dot{x} &= (A - BD^{-1}C)x + BD^{-1}y \\ u &= -D^{-1}Cx + D^{-1}y \end{aligned}$$

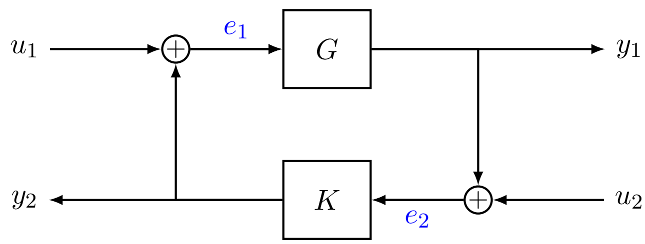
Expressed more compactly:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right]$$

Note that if we use $-x$ as the state instead of x, we will obtain the equivalent realization:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]$$

3. MIMO maps [10 points]. Consider the following interconnection of LTI systems $G(s)$ and $K(s)$:



Find the transfer matrix $H(s)$ such that: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

Note: treat $G(s)$ and $K(s)$ as matrices.

SOLUTION: Define the signals e_1 and e_2 as illustrated in the figure above. We can now write equations for the two junctions and the two system:

$$\begin{aligned} e_1 &= u_1 + y_2 \\ e_2 &= u_2 + y_1 \\ y_1 &= Ge_1 \\ y_2 &= Ke_2 \end{aligned}$$

We'd like to express y_1 as a function of u_1 and u_2. Start with the second-last equation and substitute:

$$y_1 = Ge_1 = G(u_1 + y_2) = G(u_1 + Ke_2) = G(u_1 + K(u_2 + y_1))$$

Collecting the y_1 terms and solving, we obtain:

$$y_1 = (I - GK)^{-1} (Gu_1 + GKu_2) \tag{1}$$

Similarly, we can start with the last equation and substitute

$$y_2 = Ke_2 = K(u_2 + y_1) = K(u_2 + Ge_1) = K(u_2 + G(u_1 + y_2))$$

Collecting the y_1 terms and solving, we obtain:

$$y_2 = (I - KG)^{-1} (KGu_1 + Ku_2) \tag{2}$$

Putting (1) and (2) together, we obtain the desired result:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} (I - GK)^{-1}G & (I - GK)^{-1}GK \\ (I - KG)^{-1}KG & (I - KG)^{-1}K \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Many other (equivalent) expressions for the solution are possible depending on the order and manner in which variables are eliminated. For example, the H_{11} term has several different equivalent expressions:

$$H_{11} = (I - GK)^{-1}G = G(I - KG)^{-1} = G + GK(I - GK)^{-1}G = G + G(I - KG)^{-1}KG$$

4. Mini LQR [10 points]. Consider the dynamical system $\dot{x}(t) = -x(t) + u(t)$ where $x(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$. Consider the state-feedback control law that minimizes the cost

$$\int_0^\infty \left(x(t)^2 + \mu\, u(t)^2\right)\,\mathrm{d}t.$$

Here, $\mu > 0$ is a parameter. Compute the closed-loop dynamics for the optimal controller (which depends on μ), and verify that they are stable for all values of $\mu > 0$.

Hint: The following may be useful: if (A, B) is stabilizable and (A, Q) is detectable, the Algebraic Riccati Equation $A^\mathsf{T} P + PA + Q - PBR^{-1}B^\mathsf{T} P = 0$ has a unique solution P that is symmetric and positive definite. Moreover, this P is *stabilizing*; i.e. $A + BK$ is stable where $K = -R^{-1}B^\mathsf{T} P$. Also, $u(t) = Kx(t)$ is the solution to the LQR problem with cost $\int_0^\infty (x^\mathsf{T} Qx + u^\mathsf{T} Ru)\,\mathrm{d}t$.

SOLUTION: We can directly apply the general result to this mini example. In this case, we have:

$$A = -1, \qquad B = 1, \qquad Q = 1, \qquad R = \mu.$$

The ARE becomes a scalar equation in the variable $P = p \in \mathbb{R}$. Writing it out:

$$-2p + 1 - \frac{1}{\mu}p^2 = 0$$

This is the quadratic equation: $p^2 + 2\mu p - \mu = 0$. Solving yields:

$$p = -\mu \pm \sqrt{\mu^2 + \mu}$$

Notice that when $\mu > 0$, one of these roots is positive and the other is negative. Since we want the positive definite solution (i.e. $p > 0$), we choose the positive root. The closed-loop eigenvalues are:

$$\begin{aligned} A + BK &= A - BR^{-1}B^\mathsf{T} P \\ &= (-1) - \frac{1}{\mu}\left(-\mu + \sqrt{\mu^2 + \mu}\right) \\ &= -\frac{\sqrt{\mu^2 + \mu}}{\mu} \\ &= -\sqrt{1 + \frac{1}{\mu}} \end{aligned}$$

Therefore, the closed-loop dynamics are:

$$\dot{x}(t) = -\left(\sqrt{1 + \frac{1}{\mu}}\right) x(t)$$

The coefficient multiplying $x(t)$ is negative for all $\mu > 0$, so the closed-loop dynamics are stable.

As $\mu \to 0$, we place relatively more weight on x being small. The closed-loop A matrix tends to $-\sqrt{1 + \frac{1}{\mu}} \to -\infty$. In other words, the dynamics get faster, driving $x \to 0$ more quickly. As $\mu \to \infty$, we place relatively more weight on u being small. The closed-loop A matrix tends to $-\sqrt{1 + \frac{1}{\mu}} \to -1$. In other words, in we obtain $K = 0$, i.e. $u = 0$, so we use no control effort at all.

5. Imaginary eigenvalues of the Hamiltonian [BONUS: up to 10 points]. Consider the matrix

$$H = \begin{bmatrix} A & -BR^{-1}B^\mathsf{T} \\ -C^\mathsf{T}C & -A^\mathsf{T} \end{bmatrix}$$

where $R = R^\mathsf{T} \succ 0$. Prove that if (A, B) is controllable and (A, C) is observable, then H does not have any eigenvalues on the imaginary axis.

Note: This is a difficult problem. I will also be very strict in grading this problem, so I recommend solving the other problems first!

SOLUTION:

We will prove the contrapositive: if H has an eigenvalue on the imaginary axis, then either (A, B) is uncontrollable or (A, C) is unobservable.

Let $\lambda = j\omega$ be a purely imaginary eigenvalue ($\omega \in \mathbb{R}$) and let $\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$ be an associated (nonzero) eigenvector. Then we may write:

$$\begin{bmatrix} A & -BR^{-1}B^\mathsf{T} \\ -C^\mathsf{T}C & -A^\mathsf{T} \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = j\omega \begin{bmatrix} v \\ w \end{bmatrix}$$

Expanding and rearranging the equations, we obtain:

$$\begin{aligned} (A - j\omega I)v &= BR^{-1}B^\mathsf{T}w \\ (A^\mathsf{T} + j\omega I)w &= -C^\mathsf{T}Cv \end{aligned}$$

The key here is to observe that the matrices on the left-hand sides of these equations are conjugate transposes of one another. Take the conjugate transpose of the second equation and obtain:

$$(A - j\omega I)v = BR^{-1}B^\mathsf{T}w \tag{3a}$$
$$w^*(A - j\omega I) = -v^*C^\mathsf{T}C \tag{3b}$$

Now let's do what we can to make the left-hand sides cancel. Multiply the first equation on the left by w^* and the second equation on the right by v, and obtain:

$$w^*BR^{-1}B^\mathsf{T}w = w^*(A - j\omega I)v = -v^*C^\mathsf{T}Cv$$

In other words, $w^*BR^{-1}B^\mathsf{T}w + v^*C^\mathsf{T}Cv = 0$. But these are sign-definite matrices! i.e. $BR^{-1}B^\mathsf{T} \succeq 0$ and $C^\mathsf{T}C \succeq 0$. The only way we can have $w^*BR^{-1}B^\mathsf{T}w + v^*C^\mathsf{T}Cv = 0$ is if both terms are zero. Therefore:

$$\begin{aligned} v^*C^\mathsf{T}Cv = 0 \quad &\Longrightarrow \quad \|Cv\|^2 = 0 \quad \Longrightarrow \quad Cv = 0 \\ w^*BR^{-1}B^\mathsf{T}w = 0 \quad &\Longrightarrow \quad B^\mathsf{T}w = 0 \quad \Longrightarrow \quad w^*B = 0 \end{aligned}$$

Substituting these results back into (3), we conclude that:

$$(A - j\omega I)v = 0 \quad \text{and} \quad Cv = 0 \tag{4a}$$
$$w^*(A - j\omega I) = 0 \quad \text{and} \quad w^*B = 0 \tag{4b}$$

The eigenvector $\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$ is nonzero by assumption, so either $v \neq 0$ or $w \neq 0$. If $v \neq 0$, then (4a) implies that (A, C) is unobservable by the PBH observability test. Similarly, if $w \neq 0$, then (4b) implies that (A, B) is uncontrollable by the PBH controllability test.

Therefore, either (A, B) is uncontrollable or (A, C) is unobservable, as required.