ECE 717
Exam 3 – Fall 2019

Name: ________________________________

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

____________ **(total score)**

1. Minimal realizations [10 points]. Consider the system with state-space realization:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

a) Show that this realization is not minimal.

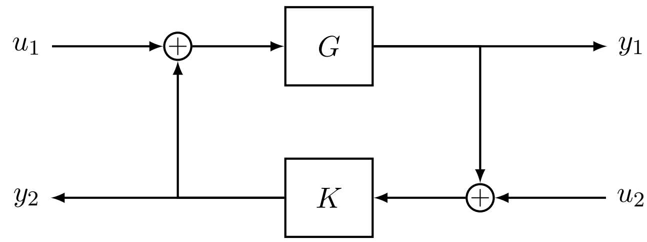
b) Find a minimal realization for this system.

2. Inverting an LTI system [10 points]. Consider the LTI system governed by the dynamics:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, and $D \in \mathbb{R}^{m \times m}$ is an invertible matrix. Note that the system is *square* (same number of inputs as outputs). Find a state-space realization for the *inverse* of this system. The *inverse* is an LTI system where the input and output are swapped (the input is y and the output is u) and the same dynamics as above are satisfied.

3. MIMO maps [10 points]. Consider the following interconnection of LTI systems $G(s)$ and $K(s)$:



Find the transfer matrix $H(s)$ such that: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

Note: treat $G(s)$ and $K(s)$ as matrices.

4. Mini LQR [10 points]. Consider the dynamical system $\dot{x}(t) = -x(t) + u(t)$ where $x(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$. Consider the state-feedback control law that minimizes the cost

$$\int_0^\infty \left(x(t)^2 + \mu\, u(t)^2\right)\, \mathrm{d}t.$$

Here, $\mu > 0$ is a parameter. Compute the closed-loop dynamics for the optimal controller (which depends on μ), and verify that they are stable for all values of $\mu > 0$.

Hint: The following may be useful: if (A, B) is stabilizable and (A, Q) is detectable, the Algebraic Riccati Equation $A^\mathsf{T} P + PA + Q - PBR^{-1}B^\mathsf{T} P = 0$ has a unique solution P that is symmetric and positive definite. Moreover, this P is *stabilizing*; i.e. $A + BK$ is stable where $K = -R^{-1}B^\mathsf{T} P$. Also, $u(t) = Kx(t)$ is the solution to the LQR problem with cost $\int_0^\infty (x^\mathsf{T} Qx + u^\mathsf{T} Ru)\, \mathrm{d}t$.

5. Imaginary eigenvalues of the Hamiltonian [BONUS: up to 10 points]. Consider the matrix

$$H = \begin{bmatrix} A & -BR^{-1}B^{\mathsf{T}} \\ -C^{\mathsf{T}}C & -A^{\mathsf{T}} \end{bmatrix}$$

where $R = R^{\mathsf{T}} \succ 0$. Prove that if (A, B) is controllable and (A, C) is observable, then H does not have any eigenvalues on the imaginary axis.

Note: This is a difficult problem. I will also be very strict in grading this problem, so I recommend solving the other problems first!