ECE 717
Exam 2 – Fall 2019

SOLUTIONS

Name: ____________________________________

1. _____________

2. _____________

3. _____________

4. _____________

_____________ **(total score)**

1. True or False [18 points]. For each question, circle TRUE or FALSE. No explanation is required.

a) If two realizations for the same transfer function have the same number of states (A is the same size for both), then there must exist a state transformation matrix T that transforms one realization into the other (and vice versa), even if the realizations are not minimal. TRUE **[FALSE]**

SOLUTION: (False) This is only true for minimal realizations.

b) If a linear time-invariant system is BIBO stable, then it is always Lyapunov stable. TRUE **[FALSE]**

SOLUTION: (False) This is only true for minimal realizations.

c) If a linear time-invariant system is asymptotically stable, then it is always exponentially stable as well. **[TRUE]** FALSE

SOLUTION: (True) For linear systems, asymptotic stability implies exponential stability.

d) When using the input $u(t) = \sin(t)$, the output $y(t)$ of a system grows unbounded with time. We can conclude that the system is definitely NOT BIBO stable. **[TRUE]** FALSE

SOLUTION: (True) BIBO stability means *all* bounded inputs produce bounded outputs. One example of this failing to occur means that the system is not BIBO stable

e) When using the input $u(t) = \sin(t)$, the output $y(t)$ of the system is bounded. We can conclude that system is definitely BIBO stable. TRUE **[FALSE]**

SOLUTION: (False) Showing that one bounded input produces a bounded output does not necessarily mean that *all* bounded inputs produce bounded outputs.

f) If $\dot{x}(t) = Ax(t)$ is asymptotically stable, then so is $x[k+1] = Ax[k]$ (same A matrix for both systems). TRUE **[FALSE]**

SOLUTION: (False) The stability requirements are different in continuous and discrete time. In continuous time, we require $\text{Re}(\lambda) < 0$ while in discrete time, we require $|\lambda| < 1$.

2. Minimal realizations [12 points].

a) Consider the following state-space realization:

$$\left[\begin{array}{ccccc|cc} 3 & 0 & 0 & 3 & 0 & 0 & 7 \\ 3 & 2 & 7 & 2 & 8 & 1 & 0 \\ 0 & 3 & 1 & 2 & 1 & 2 & 3 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Write down a minimal realization for this system.

SOLUTION: The last two states are not affected by any other states, nor are they affected by the input. Therefore these states are *uncontrollable*. Removing them, we are left with:

$$\left[\begin{array}{ccc|cc} 3 & 0 & 0 & 0 & 7 \\ 3 & 2 & 7 & 1 & 0 \\ 0 & 3 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last two states in this realization do not affect any other states, nor do they affect the output. Therefore, these states are *unobservable*. Removing them, we are left with:

$$\left[\begin{array}{c|cc} 3 & 0 & 7 \\ \hline 0 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right]$$

This realization only has one state and $B \neq 0$ and $C \neq 0$, so it cannot be reduced any further; it is minimal.

b) Find a minimal realization for a system whose impulse response is $h(t) = 2e^{-t} - 3e^{-2t}$.

SOLUTION: We can use linearity. A system with impulse response e^{at} has realization:

$$\left[\begin{array}{c|c} a & 1 \\ \hline 1 & 0 \end{array}\right]$$

Therefore, a system with impulse response $2e^{-t} - 3e^{-2t}$ has realization:

$$\left[\begin{array}{c|c} -1 & 2 \\ \hline 1 & 0 \end{array}\right] + \left[\begin{array}{c|c} -2 & -3 \\ \hline 1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 0 & 2 \\ 0 & -2 & -3 \\ \hline 1 & 1 & 0 \end{array}\right]$$

and it's clear that this realization is minimal.

ALT. SOLUTION: An alternate approach is to compute the transfer function by taking the Laplace transform. Then we obtain:

$$G(s) = \frac{2}{s+1} - \frac{3}{s+2} = \frac{-s+1}{(s+1)(s+2)} = \frac{-s+1}{s^2+3s+2}$$

Then we can find minimal realization using the controllable canonical form:

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & -3 & 1 \\ \hline 1 & -1 & 0 \end{array}\right]$$

3. Lyapunov stability for a nonlinear system [10 points]. Consider the scalar nonlinear dynamical system:

$$\dot{x}(t) = -x(t) + \frac{1}{4}x(t)^3, \qquad x(0) = x_0$$

Prove that the point $\tilde{x} = 0$ is a locally stable equilibrium point.
Hint: use the Lyapunov function $V(x) = x^2$.

SOLUTION: We will use the Lyapunov function $V(x) = x^2$ to prove stability. This amounts to showing that $V(x(t))$ is a decreasing function of t. We can ascertain this by evaluating the time derivative:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}x(t)^2 \\ &= 2x(t)\dot{x}(t) \\ &= 2x(t)\left(-x(t) + \frac{1}{4}x(t)^3\right) \\ &= -\frac{1}{2}x(t)^2\left(4 - x(t)^2\right) \end{aligned}$$

Whenever $-2 < x(t) < 2$, the expression above is negative, i.e. V is decreasing. But if V is decreasing, then so is $|x(t)|$. So if $|x_0| < 2$, then $|x(t)| < 2$ for all $t \geq 0$ as well. In other words, the point $x = 0$ is a locally stable equilibrium point.

Note that this Lyapunov function does not prove global stability, because if $|x(t)| > 2$, then V is not decreasing.

ALT. SOLUTION: The solution above is based on *Lyapunov's direct method*, which is to pick a Lyapunov candidate and directly show that it satisfies the definition of a Lyapunov function in some local neighborhood of the equilibrium point.

Another way to solve the problem is to use *Lyapunov's indirect method*. We did not cover this in class, so I was not expecting anybody to do this, but it's nonetheless valid and some students used it, so I accepted it as a valid solution. Lyapunov's indirect method states that if the linearized dynamics (linearized about the equilibrium point) are stable, then the nonlinear system is locally asymptotically stable and we can find a quadratic Lyapunov function that certifies it.

So in this case, the linearized dynamics are:

$$\dot{x}(t) = \frac{\partial}{\partial x}\left(-x + \frac{1}{4}x^3\right)_{x=0}(t) = -x(t)$$

Since the system $\dot{x}(t) = -x(t)$ is clearly stable (A-matrix is -1, which has a negative real eigenvalue), the nonlinear system is locally asymptotically stable as well.

4. Evaluating quadratic integrals [10 points]. Suppose $\dot{x}(t) = Ax(t)$ and A is Hurwitz (all eigenvalues of A have a strictly negative real part). Suppose $Q = Q^\mathsf{T}$ is a symmetric matrix. We are interested in evaluating the integral

$$J(x_0) = \int_0^\infty x(t)^\mathsf{T} Q x(t) \, \mathrm{d}t$$

Where $x(t)$ is the state of the system at time t, assuming we start at $x(0) = x_0$. Prove that the value of this integral is given by:

$$J(x_0) = x_0^\mathsf{T} P x_0$$

where P satisfies the Lyapunov equation $A^\mathsf{T} P + PA + Q = 0$.

SOLUTION: Supposed P satisfies the Lyapunov equation. Substitute the expression for Q into the integral:

$$\begin{aligned}
J(x_0) &= \int_0^\infty x(t)^\mathsf{T} Q x(t) \, \mathrm{d}t \\
&= \int_0^\infty x(t)^\mathsf{T} \left(-A^\mathsf{T} P - PA \right) x(t) \, \mathrm{d}t \\
&= -\int_0^\infty \left(\dot{x}(t)^\mathsf{T} P x(t) + x(t)^\mathsf{T} P \dot{x}(t) \right) \mathrm{d}t \\
&= -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} \left(x(t)^\mathsf{T} P x(t) \right) \mathrm{d}t \\
&= -\left[x(t)^\mathsf{T} P x(t) \right]_{t=0}^{\infty} \\
&= x_0^\mathsf{T} P x_0
\end{aligned}$$

In the last step, we used the fact that A is Hurwitz, which implies that the system is asymptotically stable, so $\lim_{t \to \infty} x(t) = 0$.