ECE 717
Exam 1 – Fall 2019

SOLUTIONS

Name: ____

(Each problem is worth 20 points)

1. _____________

2. _____________

3. _____________

4. _____________

_____________ **(total score)**

1. Short answers. For each question, give a short answer. No proof or explanation is required.

a) If a square matrix is invertible, what can you say about its eigenvalues?

SOLUTION: None of the eigenvalues are zero.

Proof: A is not invertible iff its columns are not linearly independent, which is true iff some nonzero linear combination of the columns is zero, i.e., $Av = 0$ for some $v \neq 0$. But this is equivalent to zero being an eigenvalue of A.

b) If a square matrix is diagonalizable, what can you say about its eigenvectors?

SOLUTION: The matrix has a set of eigenvectors that are linearly independent.

Proof: For each eigenvalue λ_i, we can write $Ax_i = \lambda_i x_i$ where x_i is the associated eigenvector. Assembling these equations, we get:

$$A \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Or, in other words, $AT = T\Lambda$. We can diagonalize A and write $A = T\Lambda T^{-1}$ if and only if T is invertible, which is equivalent to its columns being linearly independent, i.e., A has n linearly independent eigenvectors.

c) If a square matrix with real entries is symmetric, what can you say about its eigenvalues?

SOLUTION: All eigenvalues of the matrix are real.

Proof: Let (x, λ) be a (possibly complex) eigenpair for A. So $Ax = \lambda x$. Since A is real-valued and symmetric, we have $A^{\mathsf{T}} = A$. Now compute $x^* Ax$ in two different ways:

$$\begin{aligned} x^* Ax &= x^*(Ax) = \lambda x^* x = \lambda |x|^2 \\ &= x^* A^{\mathsf{T}} x = (Ax)^* x = (\lambda x)^* x = \lambda^* |x|^2 \end{aligned}$$

Therefore $\lambda |x|^2 = \lambda^* |x|^2$. Since $x \neq 0$, we have $\lambda = \lambda^*$, i.e., λ is a real number.

d) If a square and symmetric matrix is indefinite (neither positive definite nor negative definite), what can you say about its eigenvalues?

SOLUTION: The matrix must have at least one positive eigenvalue and at least one negative eigenvalue.

A matrix is positive definite (by definition) if all eigenvalues are positive. A matrix is negative definite (by definition) if all eigenvalues are negative. If neither of these things are true, then either at least one eigenvalue is zero, or there are both positive and negative eigenvalues. As a side note, usually people mean the latter when they say “indefinite”. I accepted both answers.

2. Transfer functions. Consider a continuous-time system with the transfer function:

$$Y(s) = \frac{1}{(s+1)(s+2)} U(s)$$

a) What is the impulse response of this system? Note: the Laplace transform of e^{-at} is $\frac{1}{s+a}$.

SOLUTION: The impulse response is the inverse Laplace transform of the transfer function.

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Therefore, we have: $h(t) = e^{-t} - e^{-2t}$.

ALT. SOLUTION: We can also use convolution, by observing that $H(s) = (\frac{1}{s+1})(\frac{1}{s+2})$. Therefore, $h(t) = e^{-t} * e^{-2t}$. So by definition,

$$\begin{aligned} h(t) &= \int_0^t e^{-\tau} e^{-2(t-\tau)} \, \mathrm{d}\tau \\ &= e^{-2t} \int_0^t e^{\tau} \, \mathrm{d}\tau \\ &= e^{-2t} \left(e^t - 1 \right) \\ &= e^{-t} - e^{-2t} \end{aligned}$$

b) Find a state-space realization for this system

SOLUTION: The transfer function is:

$$H(s) = \frac{1}{s^2 + 3s + 2}$$

So one possible realization is the Controllable Canonical Form:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & -3 & 1 \\ \hline 1 & 0 & 0 \end{array} \right]$$

ALT. SOLUTION: We can also write this as a differential equation. If $y(s) = \frac{1}{s^2+3s+2} u(s)$, then we have: $(s^2 + 3s + 2) y(s) = u(s)$. Or, in the time domain:

$$\ddot{y} + 3\dot{y} + 2y = u$$

Setting $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$, we have:

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

and this produces the same result as the Controllable Canonical Form.

3. Matrix exponentials.

a) Compute the matrix exponential e^{At}, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

SOLUTION: We make use of the definition of the matrix exponential as a power series:

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2}A^2t^2 + \frac{1}{3}A^3t^3 + cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t + \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} t^2 + \cdots \\ &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \end{aligned}$$

The series stops after two terms because $A^k = 0$ for $k \geq 2$.

b) Compute the matrix exponential e^{At}, where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

SOLUTION: Decompose A as the identity plus a remainder term:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We know that $e^{(B+C)t} = e^{Bt}e^{Ct}$ if $BC = CB$. The identity matrix commutes with everything, so we may use this identity together with the result from part (a):

$$e^{At} = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} = \begin{bmatrix} e^t & t\,e^t \\ 0 & e^t \end{bmatrix}$$

ALT. SOLUTION: Proceeding as with part (a), we notice that:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad \ldots \quad A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

It's clear that this pattern continues. The sum we're after is:

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} t^k$$

Evaluating each term separately:

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{1}{k!} t^k = e^t \\ &\sum_{k=0}^{\infty} \frac{1}{k!} k t^k = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} t^k = \sum_{k=0}^{\infty} \frac{1}{k!} t^{k+1} = t\,e^t \end{aligned}$$

Therefore,

$$e^{At} = \begin{bmatrix} e^t & t\,e^t \\ 0 & e^t \end{bmatrix}$$

4. Controllability.

a) For which values of α is the following system controllable?

$$\dot{x}(t) = \begin{bmatrix} 1 & \alpha & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

SOLUTION: The controllability matrix is:

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 2+\alpha & 2+2\alpha \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The system is controllable if and only if the controllability matrix is full-rank. We can ascertain this in numerous ways: computing the determinant, looking at the columns, looking at the rows,... Let's look at columns. The first column is always independent from the other two because of the zeros in the third entry. So we conclude that the matrix drops rank only when the second column is a multiple of the third column. This happens when $2 + \alpha = 2 + 2\alpha$, i.e. when $\alpha = 0$. So the system is controllable if and only if $\alpha \neq 0$.

b) We showed in class that if (A, B) is not controllable and its controllability matrix has rank $q < n$, we can find a state transformation matrix T such that $(A, B) \to \left(\begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ 0 & \widehat{A}_{22} \end{bmatrix}, \begin{bmatrix} \widehat{B}_1 \\ 0 \end{bmatrix} \right)$ where $\widehat{A}_{11} \in \mathbb{R}^{q \times q}$. Prove that $(\widehat{A}_{11}, \widehat{B}_1)$ is controllable.

SOLUTION: Linear state transformations preserve the rank of the controllability matrix, so the controllability matrix for $\left(\begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ 0 & \widehat{A}_{22} \end{bmatrix}, \begin{bmatrix} \widehat{B}_1 \\ 0 \end{bmatrix} \right)$ must also have rank q. Let's compute the controllability matrix explicitly:

$$\widehat{B} = \begin{bmatrix} \widehat{B}_1 \\ 0 \end{bmatrix} \qquad \widehat{A}\widehat{B} = \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ 0 & \widehat{A}_{22} \end{bmatrix} \begin{bmatrix} \widehat{B}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \widehat{A}_{11}\widehat{B}_1 \\ 0 \end{bmatrix} \qquad \cdots \qquad \widehat{A}^k\widehat{B} = \begin{bmatrix} \widehat{A}_{11}^k\widehat{B}_1 \\ 0 \end{bmatrix}$$

So the controllability matrix is given by:

$$\widehat{P} = \begin{bmatrix} \widehat{B}_1 & \widehat{A}_{11}\widehat{B}_1 & \cdots & \widehat{A}_{11}^{n-1}\widehat{B}_1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

We know that $\text{rank}(\widehat{P}) = q$ but we also know that zero rows do not contribute to rank. Therefore, we can remove them and we conclude that:

$$\text{rank}\left(\begin{bmatrix} \widehat{B}_1 & \widehat{A}_{11}\widehat{B}_1 & \cdots & \widehat{A}_{11}^{n-1}\widehat{B}_1 \end{bmatrix} \right) = q$$

But this matrix only has q rows! ($\widehat{A}_{11} \in \mathbb{R}^{q \times q}$). Therefore this matrix is full-rank. But this is also the controllability matrix for $(\widehat{A}_{11}, \widehat{B}_1)$. So we conclude that $(\widehat{A}_{11}, \widehat{B}_1)$ is controllable.