ECE 532 - Lecture 4 — subspaces and solving $Ax=b$

A **subspace** is a set $S \subseteq \mathbb{R}^n$ with the properties:

1. $0 \in S$
2. if $x, y \in S$ then $x+y \in S$
3. if $x \in S$, $\alpha \in \mathbb{R}$, then $\alpha x \in S$.

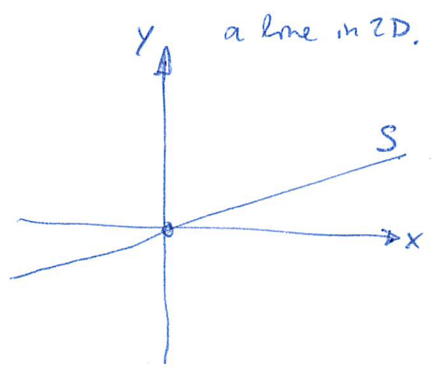
trivial examples:

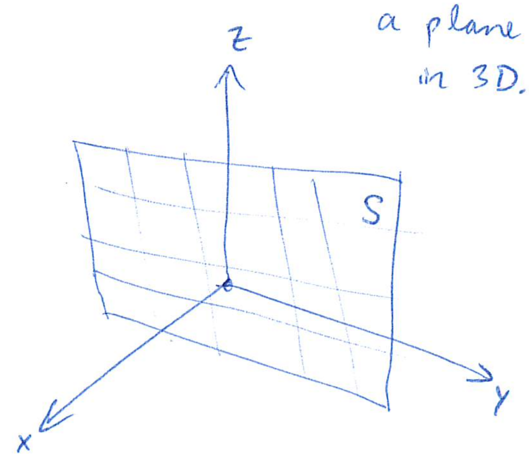
- the zero subspace $S = \{0\}$. Not to be confused with the empty set $\emptyset$ which is not a subspace!
- the entire space $S = \mathbb{R}^n$.

the span of a set of vectors is a subspace:

$$S = \text{span}(v_1, v_2, \dots, v_k).$$

☆ Subspaces look like lines, planes, (hyperplanes) that pass through the origin. Intuitive picture:





important examples

★ the **range** of a matrix $S = \text{range}(A) = R(A)$ is a subspace.

$$S = R(A) := \left\{ Ax \mid x \in \mathbb{R}^n \right\} = \text{span}(a_{\cdot 1}, \dots, a_{\cdot n}) \subseteq \mathbb{R}^m.$$

because it's simply the span of the columns of A.

★ the **nullspace** of a matrix $S = \text{null}(A) = N(A) = \text{Ker}(A)$ {a.k.a. kernel}.
is a subspace.

$$S = N(A) := \left\{ x \in \mathbb{R}^n \mid Ax = 0 \right\} \subseteq \mathbb{R}^n.$$

proof: - $A \cdot 0 = 0$ so $0 \in N(A)$.

- if $Ax = 0$ and $Ay = 0$ then $A(x+y) = 0$.
  so $x, y \in N(A) \Rightarrow x+y \in N(A)$.
- if $Ax = 0$ and $\alpha \in \mathbb{R}$, then $A(\alpha x) = 0$.
  so $x \in N(A), \alpha \in \mathbb{R} \Rightarrow \alpha x \in N(A)$. ∎

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examples.

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}. \quad \begin{cases} R(A) = \text{span} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \subseteq \mathbb{R}^3 \\ N(A) = \text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \subseteq \mathbb{R}^2 \end{cases}$$

$$A = \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 0 & 0 \end{bmatrix} \quad \begin{cases} R(A) = \text{span} \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right) \\ N(A) = \{0\} \quad (\text{zero subspace}). \end{cases}$$

☆ every subspace can be represented as the span of some set of vectors.

☆ so every subspace has a minimal representation (least # of vectors) i.e. every subspace has a basis.

☆ the number of vectors in a basis for S is called the dimension of S. Denoted dim(S).

Some observations:

- A basis is not unique.

  e.g. if $A = \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 0 & 0 \end{bmatrix}$, a basis for $R(A)$ is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right\}$.

  we could also choose $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, or $\left\{ \begin{bmatrix} 37 \\ -15 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 22 \\ 0 \end{bmatrix} \right\}$.

  $\dim(A) = 2$ in this case.

- the dimension of the zero subspace is $\dim(\{0\}) = 0$.

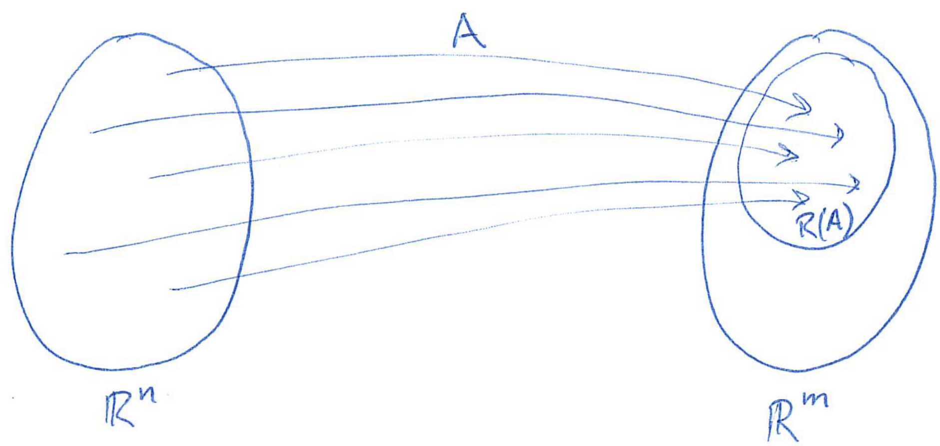
- we have the identity:

$$\dim(R(A)) = \operatorname{rank}(A).$$

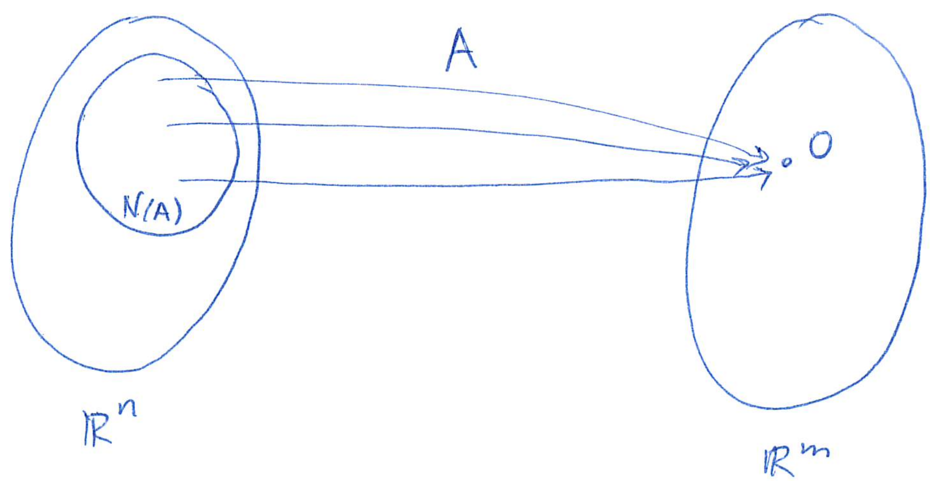
geometrical intuition

if $A \in \mathbb{R}^{m \times n}$ then it maps vectors $\mathbb{R}^n \to \mathbb{R}^m$.

☆ the range is the set of vectors in $\mathbb{R}^m$ that are reachable:



☆ the nullspace is the set of vectors that map to zero.



Fact: $\underbrace{\dim(R(A))}_{\text{rank}(A)} + \dim(N(A)) = n$

i.e. roughly, input vectors $x \in \mathbb{R}^n$ either contribute to $R(A)$, or they map to zero. All dimensions are accounted for!

(we'll prove this later!)

Linear equations : solving $Ax = b$ where $A \in \mathbb{R}^{m \times n}$.

☆ important property: if x_1 and x_2 are solutions, then $x_1 - x_2 \in N(A)$.

proof: $Ax_1 = b$ and $Ax_2 = b \Rightarrow A(x_1 - x_2) = 0$. ∎

☆ consequence: if x_1 is a solution to $Ax = b$ and x_0 is a solution to $Ax = 0$ (a.k.a. x_1 is a "particular soln" and x_0 is a "homogeneous solution") then

$x = x_1 + \alpha x_0$ is also a solution to $Ax = b$ for any $\alpha \in \mathbb{R}$.

proof: $A(x_1 + \alpha x_0) = \underbrace{Ax_1}_{b} + \alpha \underbrace{Ax_0}_{0} = b$. ∎

In fact, this implies that if $N(A) = \text{span}(n_1, \dots, n_k)$ and x_1 is one solution of $Ax = b$, then every solution of $Ax = b$ looks like:

$$x = x_1 + \alpha_1 n_1 + \alpha_2 n_2 + \dots + \alpha_k n_k \quad \text{for some choice of } \alpha_i\text{'s.}$$

more compact form: let $\{\bar{n}_1, \bar{n}_2, \dots, \bar{n}_{\bar{k}}\}$ be a basis for $N(A)$.

then let $N = [\bar{n}_1 \ \bar{n}_2 \ \cdots \ \bar{n}_{\bar{k}}] \in \mathbb{R}^{n \times \bar{k}}$.

every solution of $Ax = b$ looks like:

$$x = x_1 + N\alpha \quad \text{for some } \alpha \in \mathbb{R}^{\bar{k}}.$$

Example 1 find all solutions to $\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{b}.$

$R(A) = \mathbb{R}^2$ since $\operatorname{rank}(A) = \dim(R(A)) = 2$.

$\dim(N(A)) = 1$ since $\operatorname{rank}(A) + \dim(N(A)) = n = 3$.

1) find a particular solution: $x_1 = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$

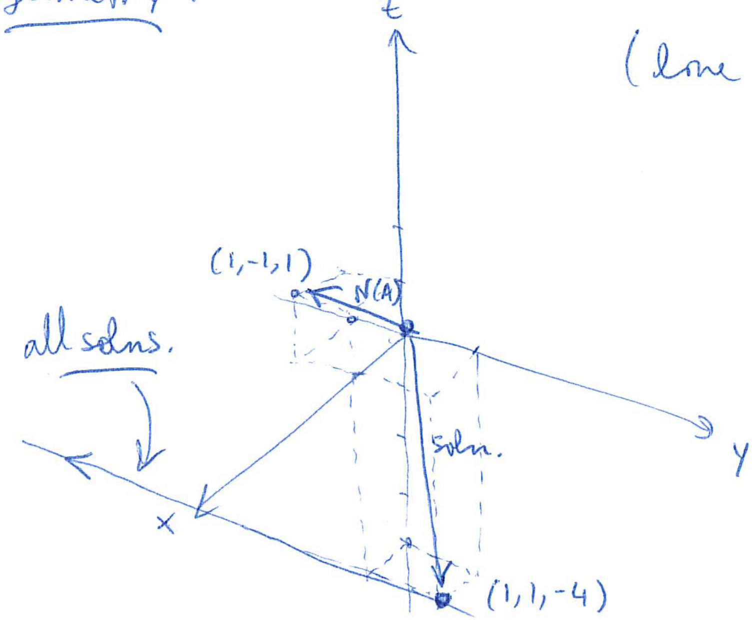
2) find a basis for $N(A)$: $N = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

3) every solution is of the form $x = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

☆ doesn't change if we use a different x_1 or N!

geometry:

(line through $\begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$)



Example 2:

$$\overbrace{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}^{b}.$$

$$\left.\begin{array}{l} R(A) = \text{span}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}. \quad (\text{rank}(A) = 1). \\ \\ N(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}. \quad (\dim N(A) = 2) \end{array}\right\} \begin{array}{c} 1+2=3 \\ (\#\text{ columns}). \end{array}$$

general solution: $x = x_1 + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$.

problem: there is no x_1! clearly $b \notin R(A)$. __No solution__.

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if we change b to $\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$ then $x_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ and every solution has the form $x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$.

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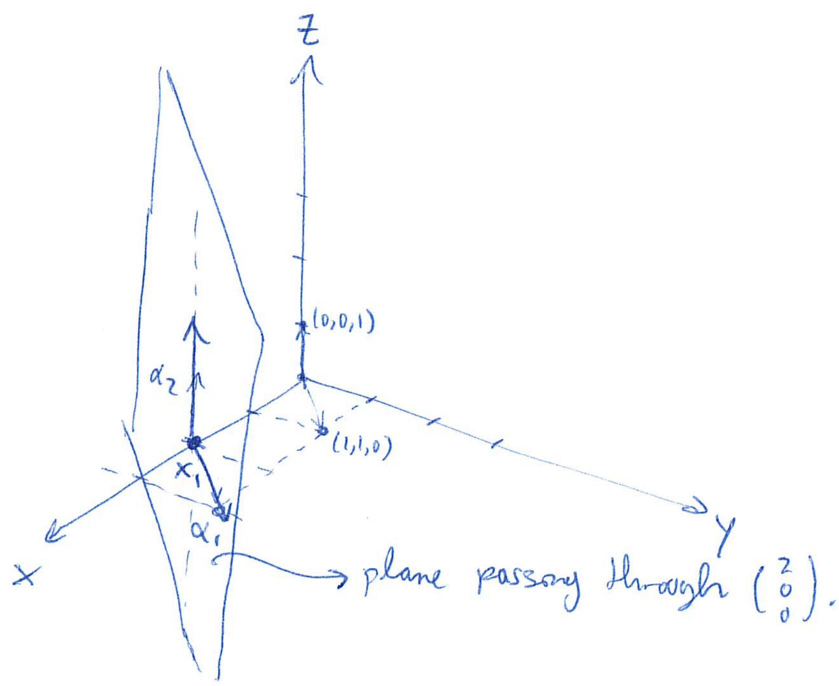

Summary:

★ set of solutions to $Ax=b$ is of the form:

$$x \in \left\{ x_1 + Nw \;\middle|\; w \in \mathbb{R}^k \right\}$$

where x_1 is any soln to $Ax=b$ and columns of N form a basis for $N(A)$. i.e. $R(N) = N(A)$.
here, $k = \dim(N(A))$.

★ this set is an affine space (a shifted subspace).

★ there are exactly three possibilities:

1) no solutions: when $b \notin R(A)$. so we can't find any x_1 such that $Ax_1 = b$

2) exactly one solution: when $k=0$ $(\text{rank}(A) = n$ and $\dim(N(A)) = 0)$ and $b \in R(A)$.

3) infinitely many solutions: when $k > 0$ and $b \in R(A)$.