4. Minimax and planning problems

- Optimizing piecewise linear functions
- Minimax problems
- Example: Chebyshev center
- Multi-period planning problems
- Example: building a house
LPs and polyhedra

Linear programs have polyhedral feasible sets:

\( \{x \mid Ax \leq b\} \implies \)

Can every polyhedron be expressed as \( Ax \leq b \)?

Not this one...
If $x, y \in \mathbb{R}^n$, then the linear combination

$$w = \alpha x + (1 - \alpha)y$$

for some $0 \leq \alpha \leq 1$

is called a **convex combination**. As we vary $\alpha$, it traces out the line segment that connects $x$ and $y$. 
LPs and polyhedra

If $Ax \leq b$ and $Ay \leq b$, and $w$ is a convex combination of $x$ and $y$, then $Aw \leq b$.

**Proof:** Suppose $w = \alpha x + (1 - \alpha)y$.

\[
Aw = A(\alpha x + (1 - \alpha)y) \\
= \alpha Ax + (1 - \alpha)Ay \\
\leq \alpha b + (1 - \alpha)b \\
= b
\]

Therefore, $Aw \leq b$, which is what we were trying to prove.

**Question:** where did we use the fact that $0 \leq \alpha \leq 1$?
The previous result implies that every polyhedron describable as $Ax \leq b$ must contain all convex combinations of its points.

- Such polyhedra are called **convex**.
- Informal definition: if you were to “shrink-wrap” it, the entire polyhedron would be covered with no extra space.

Goes the other way too: every convex polyhedron can be represented as $Ax \leq b$ for appropriately chosen $A$ and $b$. 
Some problems do not appear to be LPs but can be converted to LPs using a suitable transformation.

An important case: **convex piecewise linear functions**.

Consider the following **nonlinear** optimization:

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to:} & \quad x \geq 0
\end{align*}
\]

Where \( f(x) \) is the function:
Piecewise linear functions

The trick is to convert the problem into **epigraph** form: add an extra decision variable $t$ and turn the cost into a constraint!

\[ \begin{align*}
\text{minimize} \quad & f(x) \\
\text{subject to:} \quad & x \geq 0 
\end{align*} \]

\[ \begin{align*}
\text{minimize} \quad & t \\
\text{subject to:} \quad & t \geq f(x) \\
& x \geq 0 
\end{align*} \]
Piecewise linear functions

This feasible set is **polyhedral**. It is the set of \((x, t)\) satisfying:

\[
\{ \ t \geq -2x + 3, \quad t \geq -\frac{1}{2}x + \frac{3}{2}, \quad t \geq 3x - 9 \ \}
\]

Equivalent linear program:

\[
\begin{align*}
\text{minimize} \quad & t \\
\text{subject to:} \quad & t \geq -2x + 3, \quad t \geq -\frac{1}{2}x + \frac{3}{2} \\
& t \geq 3x - 9, \quad x \geq 0
\end{align*}
\]
Piecewise linear functions

Epigraph trick only works if it’s a **convex polyhedron**.

This epigraph is **not a convex polyhedron** so it cannot be the feasible set of a linear program.
Minimax problems

- The maximum of several linear functions is always convex. So we can minimize it using the epigraph trick. Example:

\[ f(x) = \max_{i=1, \ldots, k} \{ a_i^T x + b_i \} \]

\[
\begin{align*}
\min_x \max_{i=1, \ldots, k} \{ a_i^T x + b_i \} & \quad \implies \quad \min_{x, t} \quad t \\
\text{s.t.} \quad t \geq a_i^T x + b_i & \quad \forall i
\end{align*}
\]
Maximin problems

- The minimum of several linear functions is always concave. So we can maximize it using the epigraph trick. Example:

\[
f(x) = \min_{i=1,\ldots,k} \{ a_ix + b_i \}
\]

\[
\begin{align*}
\max_x \min_{i=1,\ldots,k} \{ a_ix + b_i \} & \quad \Rightarrow \\
\max_{x,t} & \quad t \\
s.t. & \quad t \leq a_ix + b_i \quad \forall i
\end{align*}
\]
Minimax and Maximin problems

- A minimax problem:

\[
\min_x \max_{i=1,\ldots,k} \left\{ a_i^T x + b_i \right\}
\]

\[
\min_{x,t} t \\
\text{s.t. } t \geq a_i^T x + b_i \quad \forall i
\]

- A maximin problem:

\[
\max_x \min_{i=1,\ldots,k} \left\{ a_i^T x + b_i \right\}
\]

\[
\max_{x,t} t \\
\text{s.t. } t \leq a_i^T x + b_i \quad \forall i
\]

Note: Sometimes called \textit{minmax}, \textit{min}-\textit{max}, \textit{min}/\textit{max}. Of course, \textit{minmax} \neq \textit{maxmin}!
Minimax and Maximin problems

Practical scenario:

- Paintco produces specialized paints and we are planning production for the coming year. They have some flexibility in how they produce the paints, but ultimately they require employees, as well as electricity, water, and certain chemicals.

- Nobody knows for sure how much paints will sell for, and the future price of electricity, water, and the chemicals is also unknown. But planning decision must be made now.

- Three consulting firms are hired to forecast the costs for the coming year. The three firms return with three different forecasts (cost functions $f_1, f_2, f_3$). Which one should be used?

- The risk-averse approach is to solve the minimax problem:

$$\min_x \max_{i=1,2,3} \{f_i(x)\} \neq \max_{i=1,2,3} \{\min_x f_i(x)\}$$
Absolute values

- Absolute values are piecewise linear!

\[
\min_x |x| \quad \text{s.t.} \quad Ax \leq b
\]

\[
\min_{x,t} t \quad \text{s.t.} \quad Ax \leq b \quad t \geq x \quad t \geq -x
\]

- So are sums of absolute values:

\[
\min_{x,y} |x| + |y| \quad \text{s.t.} \quad Ax \leq b
\]

\[
\min_{x,y,t,r} t + r \quad \text{s.t.} \quad t \geq x, \quad t \geq -x \quad r \geq y, \quad r \geq -y
\]
What is the largest sphere you can fit inside a polyhedron?

If $y$ is the center, then draw perpendicular lines to each face of the polyhedron.

We want to maximize the smallest $d_i$. In other words,

$$\max_y \min_{i=1,\ldots,5} d_i(y)$$

(the $y$ shown here is obviously not optimal!)
Chebyshev center

What is the largest sphere you can fit inside a polyhedron?

If \( y \) is the center, then draw perpendicular lines to each face of the polyhedron.

We want to maximize the smallest \( d_i \). In other words,

\[
\max_y \min_{i=1,\ldots,5} d_i(y)
\]

The optimal \( y \) is the Chebyshev center.
Chebyshev center

Finding the Chebyshev center amounts to solving an LP!

To compute the distance between \( y \) and the hyperplane \( a^T x = b \), notice that if the distance is \( r \), then \( y + \frac{r}{||a||} a \) belongs to the hyperplane:

\[
a^T \left( y + \frac{r}{||a||} a \right) = b
\]

Simplifying, we obtain: \( a^T y + ||a|| r = b \)

“The distance between \( y \) and each hyperplane is at least \( r \)” is equivalent to saying that \( a_i^T y + ||a_i|| r \leq b_i \) for each \( i \).
Chebyshev center

Finding the Chebyshev center amounts to solving an LP!

The transformation to an LP is given by:

\[
\begin{align*}
\max_{y} & \quad \min_{i=1,...,k} d_i(y) \\
\text{s.t.} & \quad a_i^T y \leq b_i \quad \forall i
\end{align*}
\]

\[
\Rightarrow
\begin{align*}
\max_{y,r} & \quad r \\
\text{s.t.} & \quad a_i^T y + \|a_i\| r \leq b_i \quad \forall i
\end{align*}
\]
Chebyshev center

**Example:** find the Chebyshev center of the polyhedron defined by the following inequalities:

\[ 2x - y + 2z \leq 2, \quad -x + 2y + 4z \leq 16, \quad x + 2y - 2z \leq 8, \]
\[ x \geq 0, \quad y \geq 0, \quad z \geq 0 \]

Chebyshev.ipynb
Multi-period planning problems

- Optimization problems with a **temporal** component.
- Decisions must be made over the course of multiple time periods in order to optimize an overall cost.

**Examples:**

- scheduling: classes, tasks, employees, projects,...
- sequential decisions: investments, commitments,...

The decisions at each time period are **coupled** and must be jointly optimized. Otherwise we risk making decisions that seem good at the time but end up being very costly later.
Multi-period planning problems

• These problems tend to be tricky to model. It is often not clear what the decision variables should be.

• There are often more variables than you expect.

**Important:** Decision variables aren’t always things that you decide directly!

We will see several examples of this...
Example: building a house

Several tasks must be completed in order to build a house.

- Each task takes a known amount of time to complete.
- A task may depend on other tasks, and can only be started once those tasks are complete.
- Tasks may be worked on simultaneously as long as they don’t depend on one another.
- How fast can the house be built?

Source: HBR 1963
Example: building a house

The data can be visualized using a directed graph.

- Arrows indicate task dependencies.

What are the decision variables?

- $t_i$: start time of $i^{th}$ task.
- precedence constraints are expressed in terms of $t_i$’s.
- minimize $t_x$.

Source: HBR 1963
Example: building a house

A small sample:

Let $t_l$, $t_o$, $t_m$, $t_n$, $t_t$, $t_s$ be start times of the associated tasks.

Now use the graph to write the dependency constraints:

- Tasks $o$, $m$, and $n$ can’t start until task $l$ is finished, and task $l$ takes 3 days to finish. So the constraints are:
  $$t_l + 3 \leq t_o, \quad t_l + 3 \leq t_m, \quad t_l + 3 \leq t_n$$

- Task $t$ can’t start until tasks $m$ and $n$ are finished. Therefore:
  $$t_m + 1 \leq t_t, \quad t_n + 2 \leq t_t,$$

- Task $s$ can’t start until tasks $o$ and $t$ are finished. Therefore:
  $$t_o + 3 \leq t_s, \quad t_t + 3 \leq t_s$$

Source: HBR 1963
Example: building a house

Full implementation in Julia:

House.ipynb

Follow-up: which tasks in the project are critical to finishing on time? Which tasks can withstand delays?

- related to notion of duality we will see later.