20. Logic constraints, integer variables

- If-then constraints
- Generalized assignment problems
- Logic constraints
- Modeling a restricted set of values
- Sudoku!

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If-then constraints

A single simple trick (with suitable adjustments) can help us model a great variety of if-then constraints

The trick

- We'd like to model the constraint: if $z = 0$ then $a^\mathsf{T} x \leq b$.
- Let M be an upper bound for $a^\mathsf{T} x - b$.
- Write: $a^\mathsf{T} x - b \leq Mz$
- If $z = 0$, then $a^\mathsf{T} x - b \leq 0$ as required.
  Otherwise, we get $a^\mathsf{T} x - b \leq M$, which is always true.

If-then constraints

Slight change: if $z = 1$ then $a^\mathsf{T} x \le b$

- Again, let M be an upper bound for $a^\mathsf{T} x - b$
- Write: $a^\mathsf{T} x - b \le M(1 - z)$

Reversed inequality: if $z = 0$ then $a^\mathsf{T} x \ge b$

- Write constraint as $-a^\mathsf{T} x + b \le 0$
- Let m be an upper bound on $-a^\mathsf{T} x + b$
- Write: $-a^\mathsf{T} x + b \le mz$. Same as: $a^\mathsf{T} x - b \ge -mz$
- Note: $-m$ is a *lower* bound on $a^\mathsf{T} x - b$.

If-then constraints

The converse: if $a^\mathsf{T} x \leq b$ then $z = 1$

- Equivalent to: if $z = 0$ then $a^\mathsf{T} x > b$ (contrapositive).
- The strict inequality is not really enforceable. Instead, write: if $z = 0$ then $a^\mathsf{T} x \geq b + \varepsilon$ where ε is small.
- Let m be a lower bound for $a^\mathsf{T} x - b$ and we obtain the equivalent constraint: $a^\mathsf{T} x - b \geq mz + \varepsilon(1 - z)$
- If $z = 0$, we get $a^\mathsf{T} x \geq b + \varepsilon$, as required.
  Otherwise, we get: $a^\mathsf{T} x - b \geq m$, which is always true.
- **Note:** If a, x, b are integer-valued, we may set $\varepsilon = 1$.

If-then constraints (summary)

Logic statement	Constraint
if $z = 0$ then $a^T x \le b$	$a^T x - b \le Mz$
if $z = 0$ then $a^T x \ge b$	$a^T x - b \ge mz$
if $z = 1$ then $a^T x \le b$	$a^T x - b \le M(1 - z)$
if $z = 1$ then $a^T x \ge b$	$a^T x - b \ge m(1 - z)$
if $a^T x \le b$ then $z = 1$	$a^T x - b \ge mz + \varepsilon(1 - z)$
if $a^T x \ge b$ then $z = 1$	$a^T x - b \le Mz - \varepsilon(1 - z)$
if $a^T x \le b$ then $z = 0$	$a^T x - b \ge m(1 - z) + \varepsilon z$
if $a^T x \ge b$ then $z = 0$	$a^T x - b \le M(1 - z) - \varepsilon z$

Where M and m are upper and lower bounds on $a^T x - b$.

Return to fixed costs and lower bounds

- Modeling a fixed cost: if $x > 0$ then $z = 1$.
  - Use the contrapositive: if $z = 0$ then $x \leq 0$.
  - Apply the 1^{st} rule on Slide 20-5.

- Modeling a lower bound: either $x = 0$ or $x \geq m$.
  - Equivalent to: if $x > 0$ then $x \geq m$.
  - Equivalent to the following two logical constraints: if $x > 0$ then $z = 1$, and if $z = 1$ then $x \geq m$.
  - The first one is a fixed cost (see above)
  - The second one is the 4^{th} rule on Slide 20-5.

Generalized assignment problems (GAP)

- Set of machines: $\mathcal{M} = \{1, 2, \ldots, m\}$ that can perform jobs. (think of these as the facilities in the facility problem)
- Machine i has a fixed cost of h_i if we use it at all.
- Machine i has a capacity of b_i units of work (this is new!)

- Set of jobs: $\mathcal{N} = \{1, 2, \ldots, n\}$ that must be performed. (think of these as the customers in the facility problem)
- Job j requires a_{ij} units of work to be completed if it is completed on machine i.
- Job j will cost c_{ij} if it is completed on machine i.
- Each job must be assigned to exactly one machine.

GAP model

$$
\begin{array}{llll}
\underset{x,z}{\text{minimize}} & \displaystyle\sum_{i\in\mathcal{M}} h_i z_i + \sum_{i\in\mathcal{M}}\sum_{j\in\mathcal{N}} c_{ij}x_{ij} & & \text{(fixed cost + assignment cost)} \\
\text{subject to:} & \displaystyle\sum_{i\in\mathcal{M}} x_{ij} = 1 & \forall j\in\mathcal{N} & \text{(one machine per job)} \\
 & \displaystyle\sum_{j\in\mathcal{N}} a_{ij}x_{ij} \le b_i & \forall i\in\mathcal{M} & \text{(work budget)} \\
 & x_{ij} \le z_i & \forall i\in\mathcal{M},\, j\in\mathcal{N} & \text{(if } x_{ij}>0 \text{ then } z_i=1\text{)} \\
 & x_{ij}, z_i \in \{0,1\} & \forall i\in\mathcal{M},\, j\in\mathcal{N} & \text{(all binary!)}
\end{array}
$$

- $z_i = 1$ if machine i is used, and
- $x_{ij} = 1$ if job j is performed by machine i.
- **Note:** many choices possible for M_i and aggregations.

New constraints

Let's make GAP more interesting...

1. If you use k or more machines, you must pay a penalty of λ.
2. If you operate either machine 1 or machine 2, you may not operate both machines 3 and 4 at the same time.
3. If you operate both machines 1 and 2, then machine 3 must be operated at 40% of its capacity.
4. Each job $j \in \mathcal{N}$ has a duration d_j. Minimize the time we have to wait before all jobs are completed.
   (this is called the *makespan*).

GAP 1

If you use k or more machines, you must pay a penalty of λ.

- Using k or more machines is equivalent to saying that

$$z_1 + z_2 + \cdots + z_m \geq k$$

- Let $\delta_1 = 1$ if we incur the penalty. We now have the if-then constraint: if $\sum_{i \in \mathcal{M}} z_i \geq k$ then $\delta_1 = 1$.
- Use the 6$^{\text{th}}$ rule on Slide 20-5 and obtain: $\sum_{i \in \mathcal{M}} z_i \leq m\delta_1 + (k-1)(1-\delta_1)$
- add $\lambda\delta_1$ to the cost function.

GAP 2

If you operate either machine 1 or machine 2, you may not operate both machines 3 and 4 at the same time.

- Operating machine 1 or machine 2: $z_1 + z_2 \geq 1$.
- Not operating machines 3 and 4: $z_3 + z_4 \leq 1$
- We must model $z_1 + z_2 \geq 1 \implies z_3 + z_4 \leq 1$
  - Same trick as before: model this in two steps:
    $z_1 + z_2 \geq 1 \implies \delta_2 = 1$ and $\delta_2 = 1 \implies z_3 + z_4 \leq 1$
  - First follows from 6^{th} rule on Slide 20-5
  - Second follows from 3^{rd} rule on Slide 20-5
- Result: $z_1 + z_2 \leq 2\delta_2$ and $z_3 + z_4 + \delta_2 \leq 2$.

GAP 2 (cont'd)

If you operate either machine 1 or machine 2, you may not operate both machines 3 and 4 at the same time.

We didn't do anything to ensure that when $z_i = 1$, the machines are actually operating! (we didn't explicitly disallow paying the fixed cost without using the machine).

- To force the converse as well, include the constraint: if $z_i = 1$ then $\sum_{j \in \mathcal{N}} x_{ij} \geq 1$
- Use the 4$^{\text{th}}$ rule on Slide 20-5.
- Result: $\sum_{j \in \mathcal{N}} x_{ij} \geq z_i$ (for $i = 1, 2, 3, 4$)

GAP 3

If you operate both machines 1 and 2, then machine 3 must be operated at 40% of its capacity.

- Operate both machines 1 and 2: $z_1 + z_2 \geq 2$
- Capacity of machine 3 drops: b_3 becomes $0.4b_3$.
- Two parts to the implementation:
  - $z_1 + z_2 \geq 2 \implies \delta_3 = 1$. ($6^{\text{th}}$ rule on Slide 20-5)
  - $\delta_3 = 1 \implies \sum_{j \in \mathcal{N}} a_{3j} x_{3j} \leq 0.4b_3$. ($3^{\text{rd}}$ rule on Slide 20-5)
- Equivalently, just replace b_3 by: $b_3(1 - \delta_3) + 0.4b_3\delta_3$.

GAP 4

Each job $j \in \mathcal{N}$ has a duration d_j. Minimize the time we have to wait before all jobs are completed. (the makespan)

- Machine i completes all its jobs in time: $\sum_{j \in \mathcal{N}} x_{ij} d_j$
- Minimax problem (no integer variables needed!)
- Let t be the makespan; $t = \max_{i \in \mathcal{M}} \left(\sum_{j \in \mathcal{N}} x_{ij} d_j \right)$
- Model: minimize t subject to:

$$t \geq \sum_{j \in \mathcal{N}} x_{ij} d_j \quad \text{for all } i \in \mathcal{M}$$

Logic constraints

- A **proposition** is a statement that evaluates to true or false. One example we've seen: a linear constraint $a^{\mathsf{T}}x \le b$.

- We'll use binary variables δ_i to represent propositions P_i:
$$\delta_i = \begin{cases} 1 & \text{if proposition } P_i \text{ is true} \\ 0 & \text{if proposition } P_i \text{ is false} \end{cases}$$
The term for this is that δ_i is an **indicator variable**.

How can we turn logical statements about the P_i's into algebraic statements involving the δ_i's?

Some standard notation:

$\vee$	means "or"	$\Longrightarrow$	means "implies"
$\wedge$	means "and"	$\Longleftrightarrow$	means "if and only if"
$\neg$	means "not"	$\oplus$	means "exclusive or"

Boolean algebra

Basic definitions:

P	Q	$P \wedge Q$	$P \vee Q$	$P \oplus Q$
1	1	1	1	0
1	0	0	1	1
0	1	0	1	1
0	0	0	0	0

Useful relationships:

- $\neg(P_1 \wedge \cdots \wedge P_k) = \neg P_1 \vee \cdots \vee \neg P_k$
- $\neg(P_1 \vee \cdots \vee P_k) = \neg P_1 \wedge \cdots \wedge \neg P_k$
- $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$
- $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$
- $P \oplus Q = (P \wedge \neg Q) \vee (\neg P \wedge Q)$

Logic to algebra

Statement	Constraint
$\neg P_1$	$\delta_1 = 0$
$P_1 \vee P_2$	$\delta_1 + \delta_2 \geq 1$
$P_1 \oplus P_2$	$\delta_1 + \delta_2 = 1$
$P_1 \wedge P_2$	$\delta_1 = 1,\ \delta_2 = 1$
$\neg(P_1 \vee P_2)$	$\delta_1 = 0,\ \delta_2 = 0$
$P_1 \implies P_2$	$\delta_1 \leq \delta_2$ (equivalent to: $(\neg P_1) \vee P_2$)
$P_1 \implies (\neg P_2)$	$\delta_1 + \delta_2 \leq 1$ (equivalent to: $\neg(P_1 \wedge P_2)$)
$P_1 \iff P_2$	$\delta_1 = \delta_2$
$P_1 \implies (P_2 \wedge P_3)$	$\delta_1 \leq \delta_2,\ \delta_1 \leq \delta_3$
$P_1 \implies (P_2 \vee P_3)$	$\delta_1 \leq \delta_2 + \delta_3$
$(P_1 \wedge P_2) \implies P_3$	$\delta_1 + \delta_2 \leq 1 + \delta_3$
$(P_1 \vee P_2) \implies P_3$	$\delta_1 \leq \delta_3,\ \delta_2 \leq \delta_3$
$P_1 \wedge (P_2 \vee P_3)$	$\delta_1 = 1,\ \delta_2 + \delta_3 \geq 1$
$P_1 \vee (P_2 \wedge P_3)$	$\delta_1 + \delta_2 \geq 1,\ \delta_1 + \delta_3 \geq 1$

More logic to algebra

Statement	Constraint
$P_1 \vee P_2 \vee \cdots \vee P_k$	$\sum_{i=1}^{k} \delta_i \geq 1$
$(P_1 \wedge \cdots \wedge P_k) \implies (P_{k+1} \vee \cdots \vee P_n)$	$\sum_{i=1}^{k}(1-\delta_i) + \sum_{i=k+1}^{n} \delta_i \geq 1$
at least k out of n are true	$\sum_{i=1}^{n} \delta_i \geq k$
exactly k out of n are true	$\sum_{i=1}^{n} \delta_i = k$
at most k out of n are true	$\sum_{i=1}^{n} \delta_i \leq k$
$P_n \iff (P_1 \vee \cdots \vee P_k)$	$\sum_{i=1}^{k} \delta_i \geq \delta_n,\ \delta_n \geq \delta_j,\ j = 1, \ldots, k$
$P_n \iff (P_1 \wedge \cdots \wedge P_k)$	$\delta_n + k \geq 1 + \sum_{i=1}^{k} \delta_i,\ \delta_j \geq \delta_n,\ j = 1, \ldots, k$

Modeling a restricted set of values

- We may want variable x to only take on values in the set $\{a_1, \ldots, a_m\}$.
- We introduce binary variables $y_1, \ldots, y_m$ and the constraints

$$x = \sum_{j=1}^{m} a_j y_j, \qquad \sum_{j=1}^{m} y_j = 1, \qquad y_j \in \{0, 1\}$$

- y_i serves to select which a_i will be selected.
- The set of variables $\{y_1, y_2, \ldots, y_m\}$ is called a **special ordered set** (SOS) of variables.

Example: building a warehouse

- Suppose we are modeling a facility location problem in which we must decide on the size of a warehouse to build.
- The choices of sizes and associated cost are shown below:

Size	Cost
10	100
20	180
40	320
60	450
80	600

Warehouse sizes and costs

Example: building a warehouse

- Using binary decision variables $x_1, x_2, \ldots, x_5$, we can model the cost of building the warehouse as

$$\text{cost} = 100x_1 + 180x_2 + 320x_3 + 450x_4 + 600x_5.$$

- The warehouse will have size

$$\text{size} = 10x_1 + 20x_2 + 40x_3 + 60x_4 + 80x_5,$$

- and we have the SOS constraint

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1.$$

What about integers?

- What if x is an integer, i.e. $x \in \{1, 2, \ldots, 10\}$
- First option: use 10 separate variables:

$$x = \sum_{k=1}^{10} k\, y_k, \qquad \sum_{k=1}^{10} y_k = 1, \qquad y_k \in \{0, 1\}$$

- Another option: use 4 binary variables (less symmetry):

$$x = y_1 + 2y_2 + 4y_3 + 8y_4, \quad 1 \le x \le 10, \quad y_k \in \{0, 1\}$$

Performance is solver-dependent. If the solver allows integer constraints directly, that's often the right choice.

Example: Sudoku

					1			
2	7			9		5		
	8				5			3
		8		3			2	
	5		1		2		9	
	1			5		7		
5			6				3	
		9		1			6	2
			2					

- fill grid with numbers $\{1, 2, \ldots, 9\}$
- each row and each column contains distinct numbers
- each 3×3 cluster contains distinct numbers

Example: Sudoku

- Decision variables: $X \in \{0,1\}^{9\times 9\times 9}$ (729 binary variables)

$$X_{ijk} = \begin{cases} 1 & \text{if } (i,j) \text{ entry is a } k \\ 0 & \text{otherwise} \end{cases}$$

  Can fill in known entries right away.

- Basic constraints: (324 in total)
  - $\sum_{k=1}^{9} X_{ijk} = 1 \quad \forall i,j$ (SOS constraint)
  - $\sum_{i=1}^{9} X_{ijk} = 1 \quad \forall j,k$ (column j contains exactly one k)
  - $\sum_{j=1}^{9} X_{ijk} = 1 \quad \forall i,k$ (row i contains exactly one k)
  - $\sum_{(i,j)\in C} X_{ijk} = 1 \quad \forall C,k$ (cluster C contains exactly one k)

- Much trickier to model using other integer representations!

- Julia code: Sudoku.ipynb