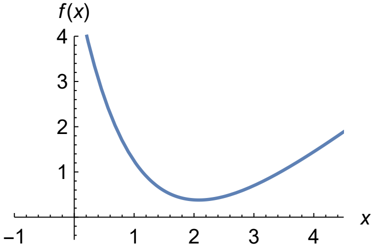
17. Intro to nonconvex models

- Overview
- Discrete models
- Mixed-integer programming
- Examples

Laurent Lessard (www.laurentlessard.com)

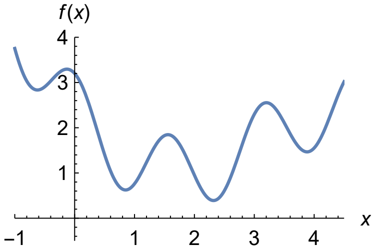
Convex programs

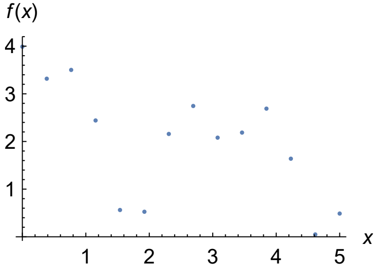
- We saw: LP, QP, QCQP, SOCP, SDP
- Can be efficiently solved
- Optimal cost can be bounded above and below
- Local optimum is global



Nonconvex programs

- In general, cannot be efficiently solved
- Cost cannot be bounded easily
- Usually we can only guarantee local optimality
- Difficulty depends strongly on the instance





Outline of the remainder of the course

- Integer (linear) programs
  - it's an LP where some or all variables are discrete (boolean, integer, or general discrete-valued)
  - If all variables are integers, it's called IP or ILP
  - If variables are mixed, it's called MIP or MILP
- Nonconvex nonlinear programs
  - If continuous, it's called NLP
  - If discrete, it's called MINLP
- Approximation and relaxation
  - Can we solve solve a convex problem instead?
  - If not, can we approximate?

Discrete variables

Why are discrete variables sometimes necessary?

1. A decision variable is fundamentally discrete

- Whether a particular power plant is used or not $\{0, 1\}$
- Number of automobiles produced $\{0, 1, 2, ...\}$
- Dollar bill amount $\{\$1, \$5, \$10, \$20, \$50, \$100\}$

Discrete variables

Why are discrete variables sometimes necessary?

2. Used to represent a logic constraint **algebraically**.

- "At most two of the three machines can run at once."

  $$z_1 + z_2 + z_3 \leq 2 \qquad (z_i \text{ is 1 if machine } i \text{ is running})$$

- "If machine 1 is running, so is machine 2."

  $$z_1 \leq z_2$$

- Goal: (logic constraint) $\iff$ (LP with extra boolean variables)

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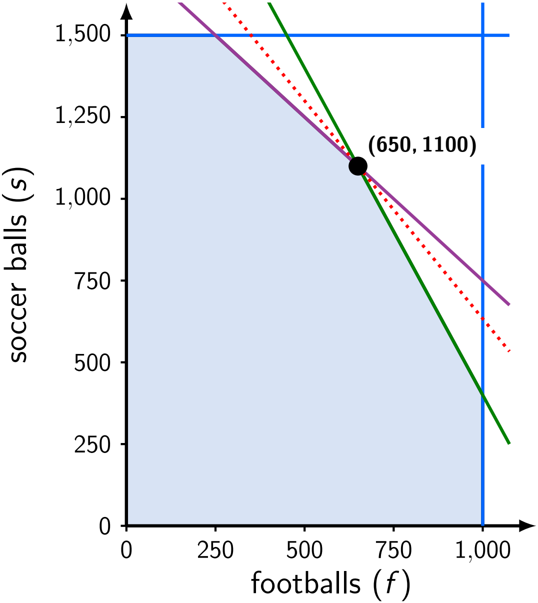

$$\max_{f,s} \quad 12f + 9s$$

$$\text{s.t.} \quad 4f + 2s \le 4800$$

$$f + s \le 1750$$

$$0 \le f \le 1000$$

$$0 \le s \le 1500$$

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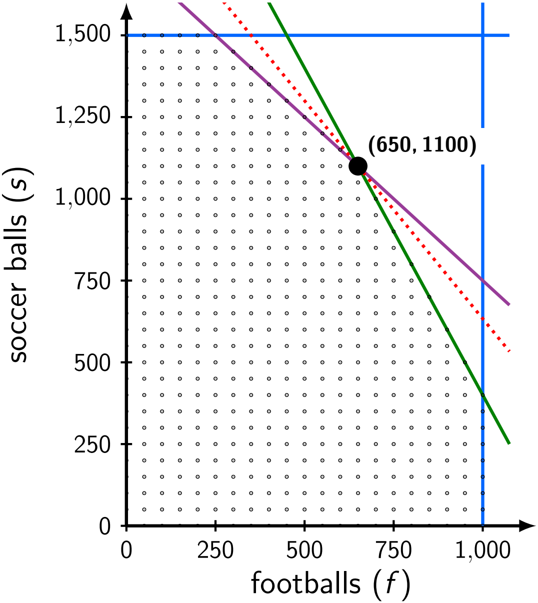

$$\max_{f,s} \quad 12f + 9s$$

$$\text{s.t.} \quad 4f + 2s \leq 4800$$

$$f + s \leq 1750$$

$$0 \leq f \leq 1000$$

$$0 \leq s \leq 1500$$

f and s are multiples of 50

Same solution!

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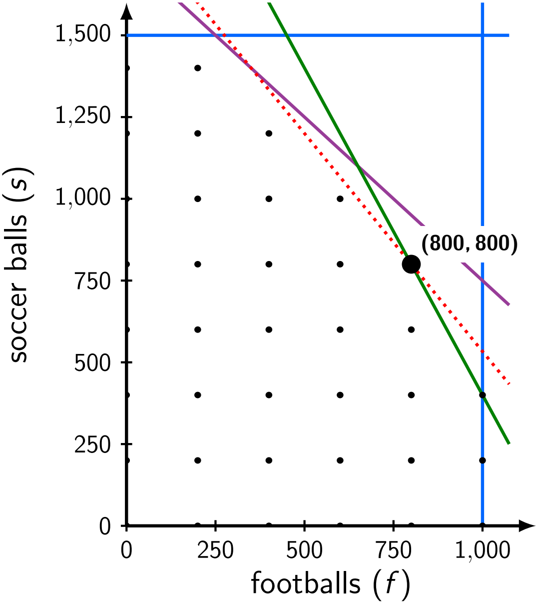

$$\max_{f,s} \quad 12f + 9s$$

$$\text{s.t.} \quad 4f + 2s \leq 4800$$

$$f + s \leq 1750$$

$$0 \leq f \leq 1000$$

$$0 \leq s \leq 1500$$

f and s are multiples of 200

Boundary solution!

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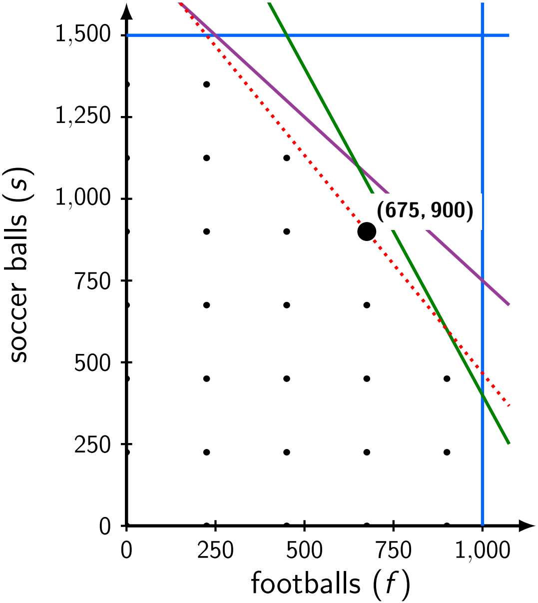

$$\max_{f,s} \quad 12f + 9s$$

$$\text{s.t.} \quad 4f + 2s \leq 4800$$

$$f + s \leq 1750$$

$$0 \leq f \leq 1000$$

$$0 \leq s \leq 1500$$

f and s are multiples of 225

Interior solution!

Mixed-integer programs

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & c^{\mathsf{T}} x \\ \text{subject to:} \quad & Ax \le b \\ & x \ge 0 \\ & x_i \in S_i \end{aligned}$$

where S_i can be:

- The real numbers, $\mathbb{R}$
- The integers, $\mathbb{Z}$
- Boolean, $\{0, 1\}$
- A discrete set, $\{v_1, v_2, \ldots, v_k\}$

Mixed-integer programs

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & c^\mathsf{T} x \\ \text{subject to:} \quad & Ax \le b \\ & x \ge 0 \\ & x_i \in S_i \end{aligned}$$

The solution can be

- Same as the LP version
- On a boundary
- In the interior (but not too far)

Common examples

- Facility location
  - locating warehouses, services, etc.
- Scheduling/sequencing
  - scheduling airline crews
- Multicommodity flows
  - transporting many different goods across a network
- Traveling salesman problems
  - routing deliveries

Knapsack problem

My knapsack holds at most 15 kg. I have the following items:

item number	**1**	**2**	**3**	**4**	**5**
weight	12 kg	2 kg	4 kg	1 kg	1 kg
value	\$4	\$2	\$10	\$2	\$1

How can I maximize the value of the items in my knapsack?

$$\text{Let } z_i = \begin{cases} 1 & \text{knapsack contains item } i \\ 0 & \text{otherwise} \end{cases}$$

Knapsack problem

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How can I maximize the value of the items in my knapsack?

$$
\begin{aligned}
\underset{z}{\text{maximize}} \quad & 4z_1 + 2z_2 + 10z_3 + 2z_4 + z_5 \\
\text{subject to:} \quad & 12z_1 + 2z_2 + 4z_3 + z_4 + z_5 \le 15 \\
& z_i \in \{0, 1\} \quad \text{for all } i
\end{aligned}
$$

notebook: Knapsack.ipynb

General (0,1) knapsack

- weights $w_1, \ldots, w_n$ and limit W.
- values $v_1, \ldots, v_n$
- decision variables $z_1, \ldots, z_n$

$$
\begin{aligned}
\underset{z}{\text{maximize}} \quad & \sum_{i=1}^{n} v_i z_i \\
\text{subject to:} \quad & \sum_{i=1}^{n} w_i z_i \le W \\
& z_i \in \{0, 1\} \quad \text{for } i = 1, \ldots, n
\end{aligned}
$$