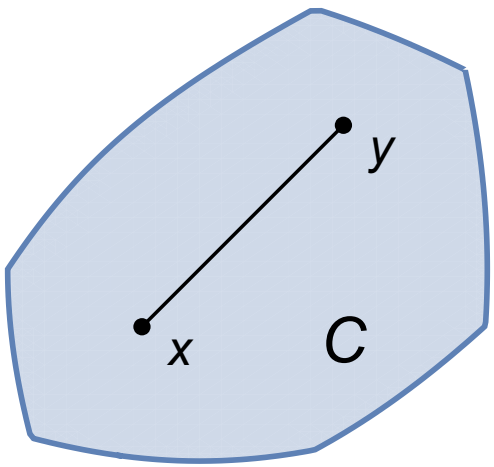
16. Review of convex optimization

- Convex sets and functions
- Convex programming models
- Network flow problems
- Least squares problems
- Regularization and tradeoffs
- Duality

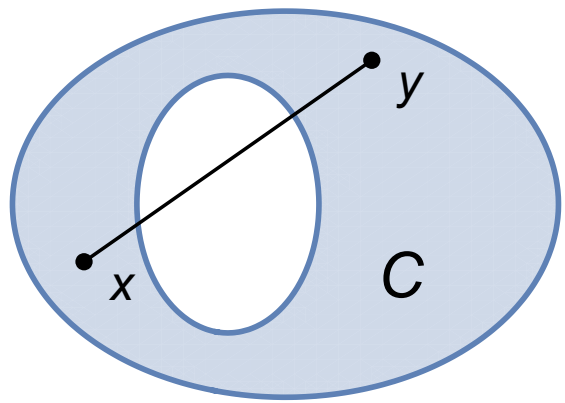
Convex sets

A set $C \subseteq \mathbb{R}^n$ is **convex** if for all $x, y \in C$ and all $0 \leq \alpha \leq 1$, we have: $\alpha x + (1 - \alpha)y \in C$.

- every line segment must be contained in the set
- can include boundary or not
- can be finite or not



convex set

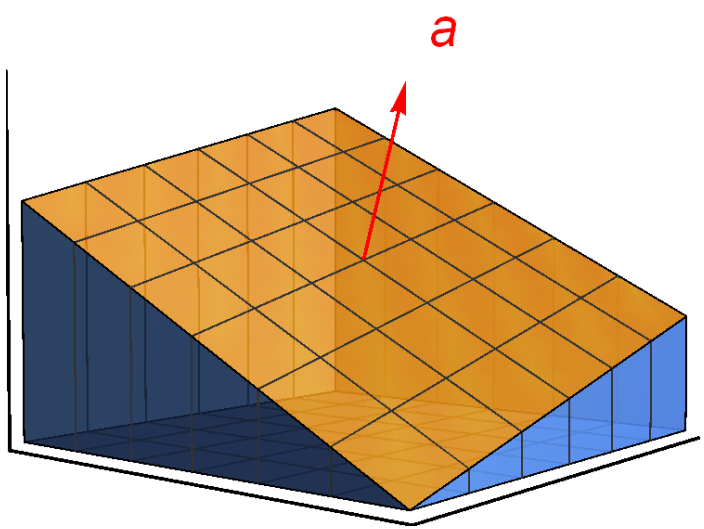


nonconvex set

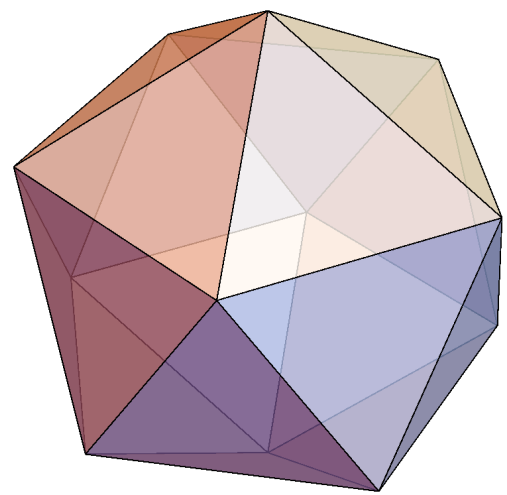
Examples

1. Polyhedron

- A linear inequality $a_i^T x \leq b_i$ is a *halfspace*.
- Intersections of halfspaces form a polyhedron: $Ax \leq b$.



Halfspace in 3D

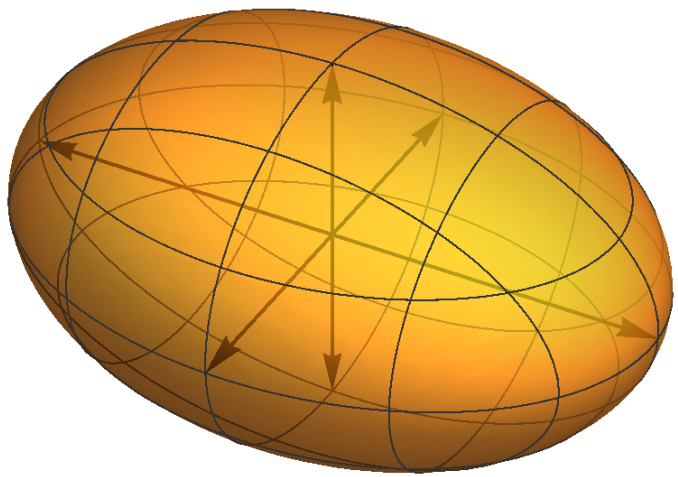


Polyhedron in 3D.

Examples

2. Ellipsoid

- A quadratic form looks like: $x^T Q x$
- If $Q \succ 0$ (positive definite; all eigenvalues positive), then the set of x satisfying $x^T Q x \leq b$ is an *ellipsoid*.

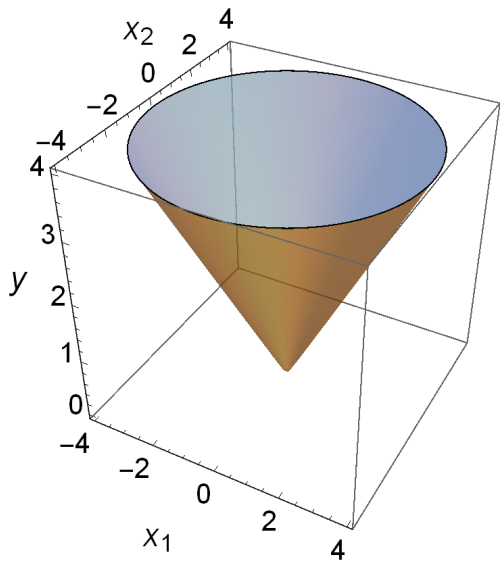


Ellipsoid

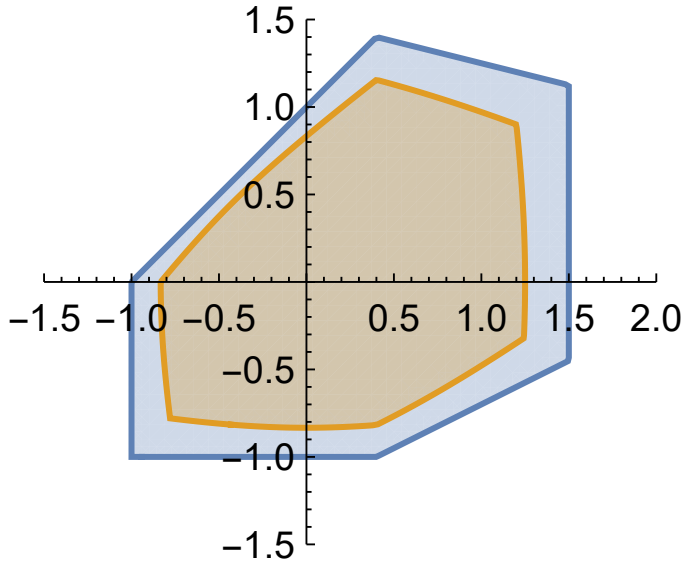
Examples

3. Second-order cone constraint

- The set of points satisfying $\|Ax + b\| \leq c^T x + d$ is called a *second-order cone constraint*.
- Example: robust linear programming



Second order cone: $\|x\| \leq y$



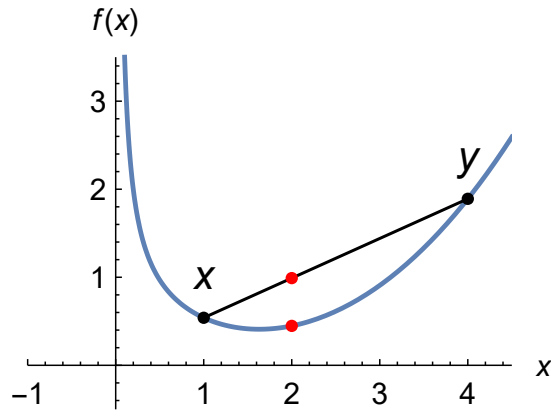
Constraints $a_i^T x + \rho\|x\| \leq b_i$

Convex functions

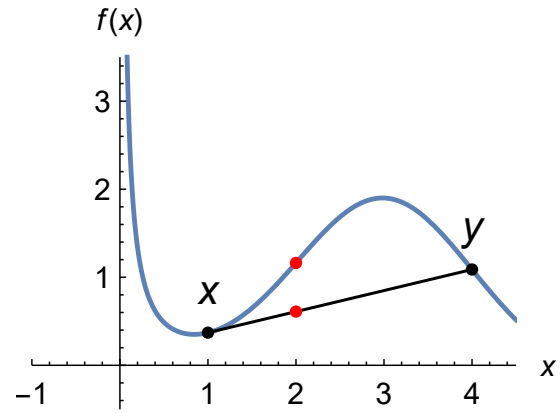
A function $f : D \to \mathbb{R}$ is a **convex function** if:

1. the domain $D \subseteq \mathbb{R}^n$ is a convex set
2. for all $x, y \in D$ and $0 \leq \alpha \leq 1$, the function f satisfies:
   $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$

- any line segment joining points of f lies above f.
- f is continuous, not necessarily smooth
- f is *concave* if $-f$ is convex.



Convex function



Nononvex function

Convex programs

$$\begin{array}{ll} \underset{x \in D}{\text{minimize}} & f_0(x) \\ \text{subject to:} & f_i(x) \leq 0 \quad \text{for } i = 1, \ldots, m \\ & h_j(x) = 0 \quad \text{for } j = 1, \ldots, r \end{array}$$

- the domain is the set D
- the cost function is f_0
- the inequality constraints are the f_i for $i = 1, \ldots, m$.
- the equality constraints are the h_j for $j = 1, \ldots, r$.
- **feasible set**: the $x \in D$ satisfying all constraints.

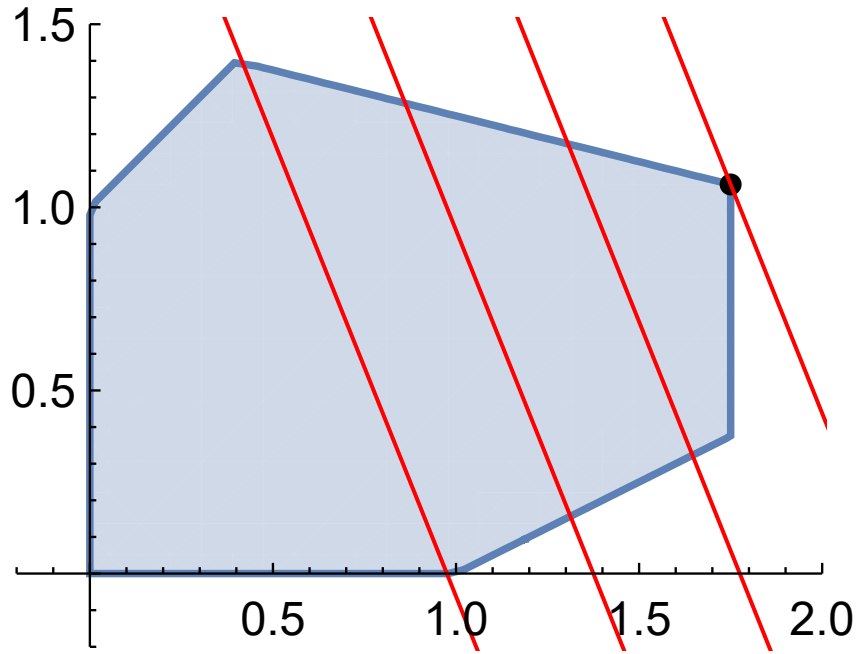
A model is **convex** if D is a convex set, all the f_i are convex functions, and the h_j are affine functions (linear + constant)

Examples

1. **Linear program (LP)**
   - cost is affine
   - all constraints are affine
   - can be maximization or minimization

Important properties

- feasible set is a polyhedron
- can be optimal, infeasible, or unbounded
- optimal point occurs at a vertex



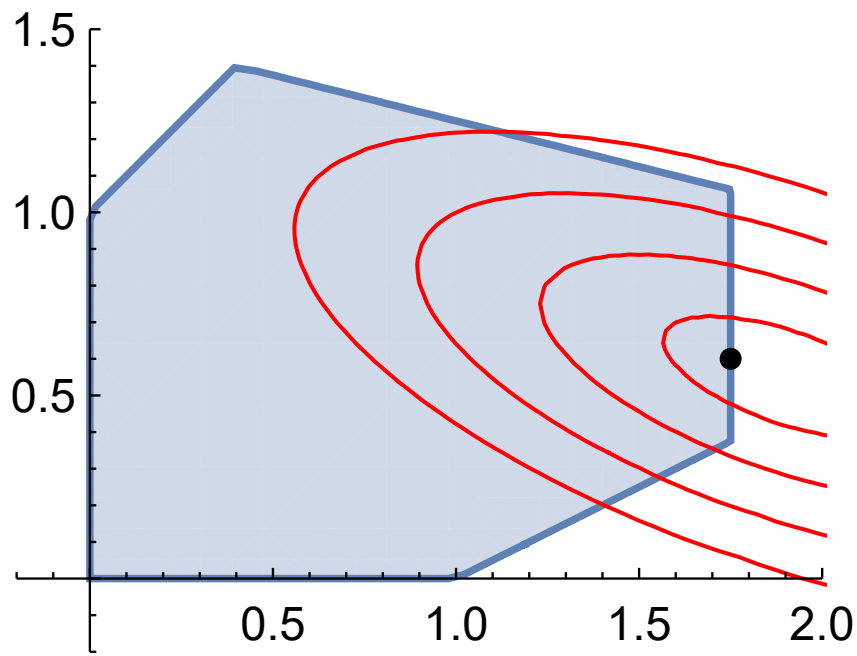
Examples

2. Convex quadratic program (QP)

- cost is a convex quadratic
- all constraints are affine
- must be a minimization

Important properties

- feasible set is a polyhedron
- optimal point occurs on boundary or in interior



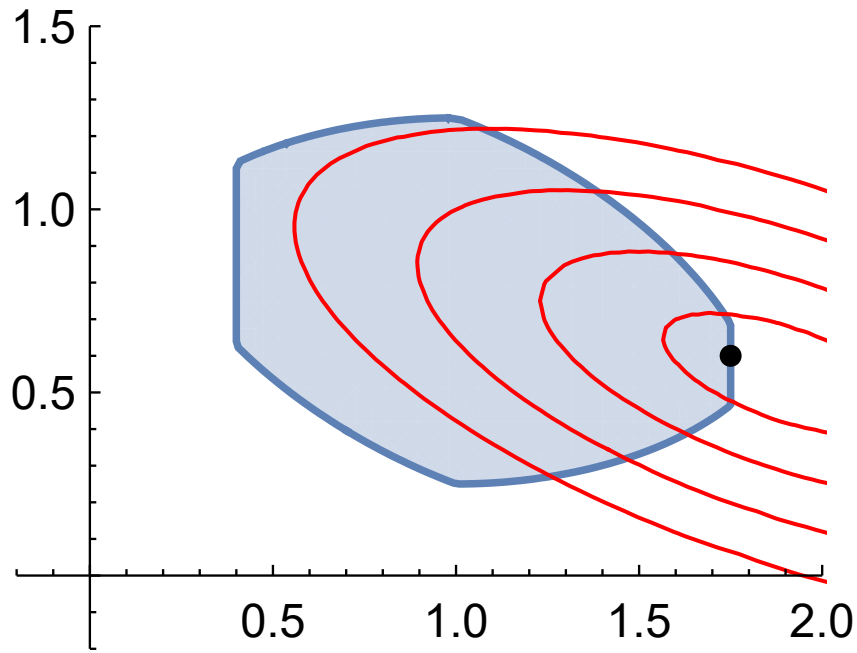
Examples

3. Convex quadratically constrained QP (QCQP)

- cost is convex quadratic
- inequality constraints are convex quadratics
- equality constraints are affine

Important properties

- feasible set is an intersection of ellipsoids
- optimal point occurs on boundary or in interior



Examples

4. Second-order cone program (SOCP)

- cost is affine
- inequality constraints are second-order cone constraints
- equality constraints are affine

Important properties

- feasible set is convex
- optimal point occurs on boundary or in interior

Hierarchy of complexity

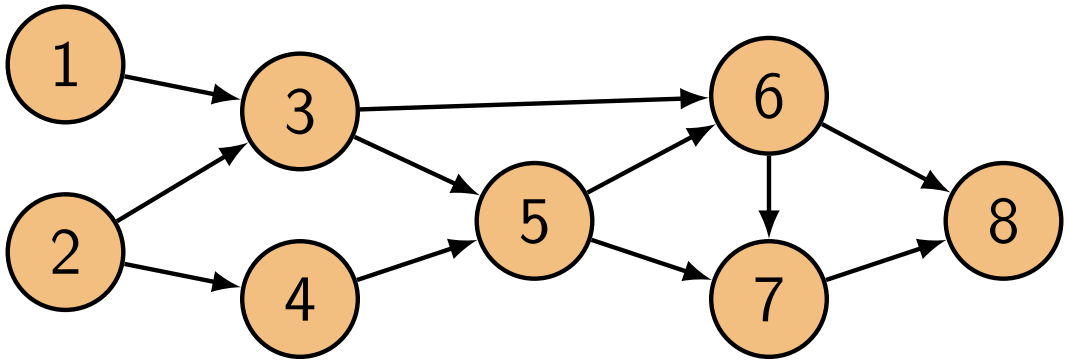
From simplest to most complicated:

1. linear program
2. convex quadratic program
3. convex quadratically constrained quadratic program
4. second-order cone program
5. semidefinite program
6. general convex program

Important notes

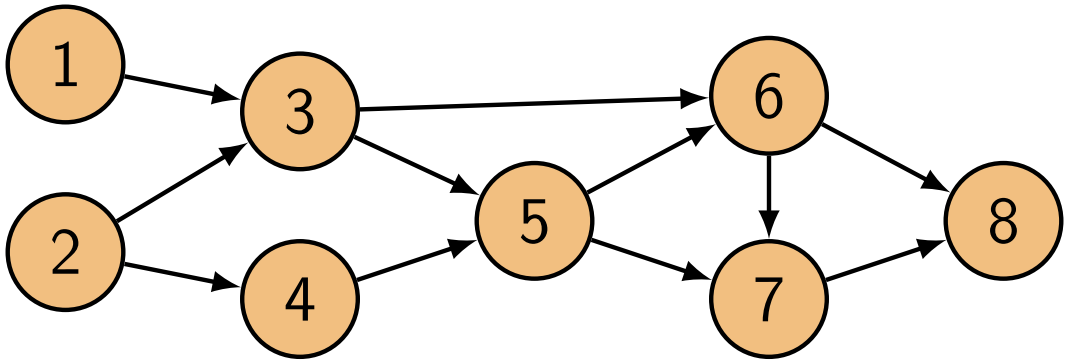
- more complicated just means that e.g. every LP is a SOCP (by setting appropriate variables to zero), but a general SOCP cannot be expressed as an LP.
- in general: strive for the simplest model possible

Network flow problems



- Each edge $(i, j) \in \mathcal{E}$ has a flow $x_{ij} \geq 0$.
- Each edge has a transportation cost c_{ij}.
- Each node $i \in \mathcal{N}$ is: a source if $b_i > 0$, a sink if $b_i < 0$, or a relay if $b_i = 0$. The sum of flows entering i must equal b_i.
- Find the flow that minimizes total transportation cost while satisfying demand at each node.

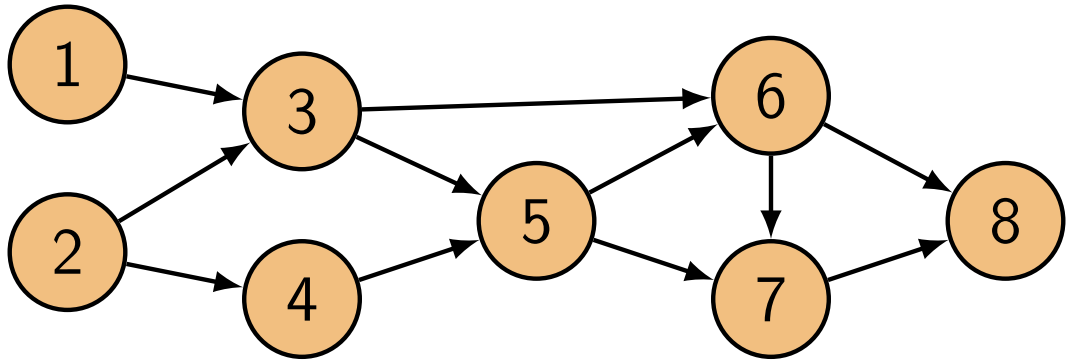
Network flow problems



- **Capacity constraints**: $p_{ij} \leq x_{ij} \leq q_{ij}$ $\quad \forall (i,j) \in \mathcal{E}.$
- **Balance constraint**: $\sum_{j \in \mathcal{N}} x_{ij} = b_i$ $\quad \forall i \in \mathcal{N}.$
- **Minimize total cost**: $\sum_{(i,j) \in \mathcal{E}} c_{ij} x_{ij}$

We assume $\sum_{i \in \mathcal{N}} b_i = 0$ (balanced graph). Otherwise, add a dummy node with no cost to balance the graph.

Network flow problems



Expanded form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{24} \\ x_{35} \\ x_{36} \\ x_{45} \\ x_{56} \\ x_{57} \\ x_{67} \\ x_{68} \\ x_{78} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \end{bmatrix}$$

$A =$ incidence matrix

Integer solutions

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^{\mathsf{T}}x \\ \text{subject to:} & Ax = b \\ & p \leq x \leq q \end{array}$$

- If A is a **totally unimodular matrix** then if demands b_i and capacities q_{ij} are integers, the flows x_{ij} are integers.
- All incidence matrices are totally unimodular.

Examples

- **Transportation problem:** each node is a source or a sink
- **Assignment problem:** transportation problem where each source has supply 1 and each sink has demand 1.
- **Transshipment problem:** like a transportation problem, but it also has relay nodes (warehouses)
- **Shortest path problem:** single source, single sink, and the edge costs are the path lengths.
- **Max-flow problem:** single source, single sink. Add a feedback path with -1 cost and minimize the cost.

Least squares

- We want to solve $Ax = b$ where $A \in \mathbb{R}^{m \times n}$.
- Typical case of interest: $m > n$ (overdetermined). If there is no solution to $Ax = b$ we try instead to have $Ax \approx b$.
- The least-squares approach: make Euclidean norm $\|Ax - b\|$ as small as possible.

Standard form:

$$\underset{x}{\text{minimize}} \quad \|Ax - b\|^2$$

It's an unconstrained convex QP.

Example: curve-fitting

- We are given noisy data points (x_i, y_i).
- We suspect they are related by $y = px^2 + qx + r$
- Find the p, q, r that best agrees with the data.

Writing all the equations:

$$\begin{aligned} y_1 &\approx px_1^2 + qx_1 + r \\ y_2 &\approx px_2^2 + qx_2 + r \\ &\vdots \\ y_m &\approx px_m^2 + qx_m + r \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \approx \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ x_m^2 & x_m & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

- Also called **regression**.

Regularization

Regularization: Additional penalty term added to the cost function to encourage a solution with desirable properties.

Regularized least squares:

$$\underset{x}{\text{minimize}} \quad \|Ax - b\|^2 + \lambda R(x)$$

- $R(x)$ is the regularizer (penalty function)
- λ is the regularization parameter
- The model has different names depending on $R(x)$.

Examples

$$\underset{x}{\text{minimize}} \quad \|Ax - b\|^2 + \lambda R(x)$$

1. If $R(x) = \|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$
   It is called: L_2 *regularization*, *Tikhonov regularization*, or *Ridge regression* depending on the application. It has the effect of **smoothing** the solution.

2. If $R(x) = \|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$
   It is called: L_1 *regularization* or *LASSO*. It has the effect of **sparsifying** the solution ($\hat{x}$ will have few nonzero entries).

3. $R(x) = \|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$
   It is called L_∞ *regularization* and it has the effect of **equalizing** the solution (makes most components equal).

Tradeoffs

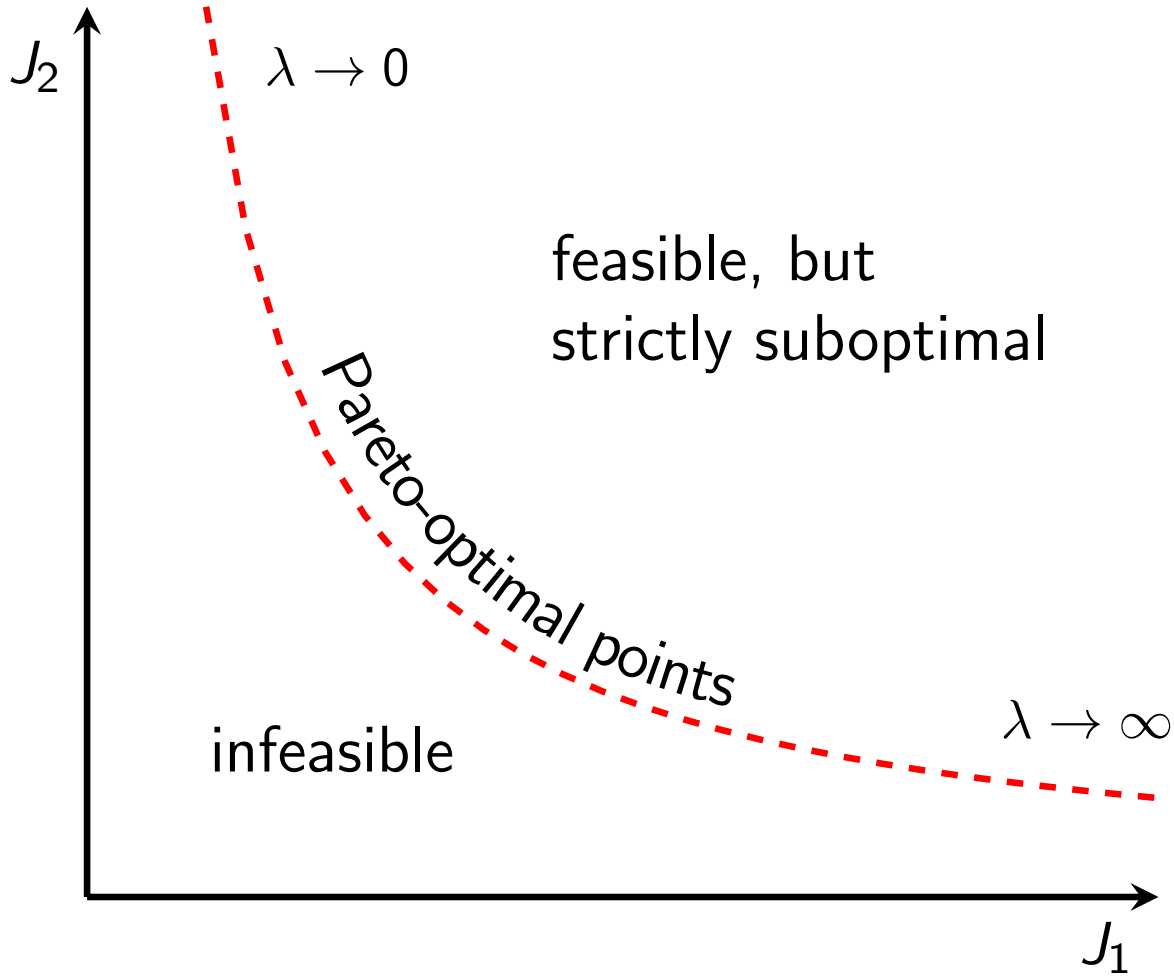
- Suppose $J_1 = \|Ax - b\|^2$ and $J_2 = \|Cx - d\|^2$.
- We would like to make **both** J_1 and J_2 small.
- A sensible approach: solve the optimization problem:

$$\underset{x}{\text{minimize}} \quad J_1 + \lambda J_2$$

  where $\lambda > 0$ is a (fixed) **tradeoff parameter**.

- Then tune λ to explore possible results.
  - When $\lambda \to 0$, we place more weight on J_1
  - When $\lambda \to \infty$, we place more weight on J_2

Pareto curve



- Pareto-optimal points can only improve in J_1 at the expense of J_2 or vice versa.

Example: Min-norm least squares

Underdetermined case: $A \in \mathbb{R}^{m \times n}$ is a wide matrix ($m \leq n$), so $Ax = b$ has infinitely many solutions.

- Look to make both $\|Ax - b\|^2$ and $\|x\|^2$ small

$$\underset{x}{\text{minimize}} \quad \|Ax - b\|^2 + \lambda \|x\|^2$$

- In the limit $\lambda \to \infty$, we get $x = 0$
- In the limit $\lambda \to 0$, we get the min-norm solution:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \|x\|^2 \\ \text{subject to:} & Ax = b \end{array}$$

Duality

Intuition: Duality is all about finding solution bounds.

- If the primal problem is a minimization, all feasible points of the primal are **upper bounds** on the optimal solution.
- The dual problem is a maximization. All feasible points of the dual are **lower bounds** on the optimal solution.

Example: LP duality

Primal problem (P)

$$\begin{array}{ll} \underset{x}{\text{maximize}} & c^T x \\ \text{subject to:} & Ax \leq b \\ & x \geq 0 \end{array}$$

Dual problem (D)

$$\begin{array}{ll} \underset{\lambda}{\text{minimize}} & b^T \lambda \\ \text{subject to:} & A^T \lambda \geq c \\ & \lambda \geq 0 \end{array}$$

If x and λ are feasible points of (P) and (D) respectively:

$$c^T x \leq p^\star \leq d^\star \leq b^T \lambda$$

- in the case of LPs, the dual of the dual is the primal

Strong duality

We have **strong duality** if $p^\star = d^\star$

- When dealing with LPs, if either the primal or dual has a finite solution, then strong duality holds.
- When dealing with general convex programs, if there is a strictly feasible point then strong duality holds. This is called **Slater's condition**.

These sorts of conditions that can guarantee strong duality are called **constraint qualifications**.

Complementary slackness

If strong duality holds, then we also have the complementary slackness property:

If the constraint $f_i(x) \leq 0$ has associated dual variable λ_i, then $f_i(x^\star)\lambda_i^\star = 0$. This means that:

- If $f_i(x^\star) < 0$ (loose constraint), then $\lambda_i^\star = 0$
- If $\lambda_i^\star > 0$ (positive dual variable), then $f_i(x^\star) = 0$

Sensitivity: The size of λ_i indicates how much a change in the constraint f_i will affect the optimal cost.