Optimal Control of Two-Player Systems with Output Feedback

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Abstract

In this article, we consider a fundamental decentralized optimal control problem, which we call the two-player problem. Two subsystems are interconnected in a nested information pattern, and output feedback controllers must be designed for each subsystem. Several special cases of this architecture have previously been solved, such as the state-feedback case or the case where the dynamics of both systems are decoupled. In this paper, we present a detailed solution to the general case. The structure of the optimal decentralized controller is reminiscent of that of the optimal centralized controller; each player must estimate the state of the system given their available information and apply static control policies to these estimates to compute the optimal controller. The previously solved cases benefit from a separation between estimation and control that allows the associated gains to be computed separately. This feature is not present in general, and some of the gains must be solved for simultaneously. We show that computing the required coupled estimation and control gains amounts to solving a small system of linear equations.

I Introduction

Many large-scale systems such as the internet, power grids, or teams of autonomous vehicles, can be viewed as a network of interconnected subsystems. A common feature of these applications is that subsystems must make control decisions with limited information. The hope is that despite the decentralized nature of the system, global performance criteria can be optimized. In this paper, we consider a particular class of decentralized control problems, illustrated in Figure 1, and develop the optimal linear controller in this framework.

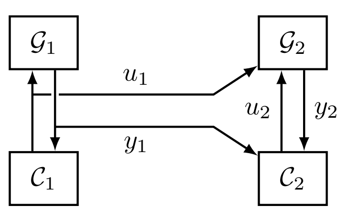

Figure 1: Decentralized interconnection

Figure 1 shows plants $\mathcal{G}_1$ and $\mathcal{G}_2$, with associated controllers $\mathcal{C}_1$ and $\mathcal{C}_2$. Controller $\mathcal{C}_1$ receives measurements only from $\mathcal{G}_1$, whereas $\mathcal{C}_2$ receives measurements from both $\mathcal{G}_1$ and $\mathcal{G}_2$. The actions of $\mathcal{C}_1$ affect both $\mathcal{G}_1$ and $\mathcal{G}_2$, whereas the actions of $\mathcal{C}_2$ only affect $\mathcal{G}_2$. In other words, information may only flow from left to right. We assume explicitly that the signal u_1 entering $\mathcal{G}_1$ is the same as the signal u_1 entering $\mathcal{G}_2$. A different modeling assumption would be to assume that these signals could be affected by separate noise.

Similarly, we assume that the measurement y_1 received by $\mathcal{C}_1$ is the same as that received by $\mathcal{C}_2$. We give a precise definition of our chosen notion of stability in Section II-B and we further discuss our choice of stabilization framework in Section III.

The feedback interconnection of Figure 1 may be represented using the linear fractional transformation shown in Figure 2.

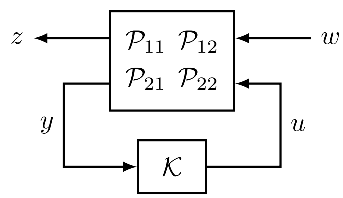

Figure 2: General four-block plant with controller in feedback

The information flow in this problem leads to a sparsity structure for both $\mathcal{P}$ and $\mathcal{K}$ in Figure 2. Specifically, we find the following block-lower-triangular sparsity pattern

$$\mathcal{P}_{22} = \begin{bmatrix} \mathcal{G}_{11} & 0 \\ \mathcal{G}_{21} & \mathcal{G}_{22} \end{bmatrix} \quad \text{and} \quad \mathcal{K} = \begin{bmatrix} \mathcal{K}_{11} & 0 \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix} \tag{1}$$

while $\mathcal{P}_{11}$, $\mathcal{P}_{12}$, $\mathcal{P}_{21}$ are full in general. We assume $\mathcal{P}$ and $\mathcal{K}$ are finite-dimensional continuous-time linear time-invariant systems. The goal is to find an $\mathcal{H}_2$-optimal controller $\mathcal{K}$ subject to the constraint (1). We paraphrase our main result, found in Theorem 6. Consider the state-space dynamics

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + w \\
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v
\end{aligned}$$

where w and v are Gaussian white noise. The objective is to minimize the quadratic cost of the standard LQG problem, subject to the constraint that

Player 1 measures y_1 and chooses u_1

Player 2 measures y_1, y_2 and chooses u_2

Here we borrow the terminology from game theory, and envisage separate *players* performing the actions u_1 and u_2. We will show that the optimal controller has the form

$$u = K x_{|y_1,u_\zeta} + \begin{bmatrix} 0 & 0 \\ H & J \end{bmatrix} \left(x_{|y,u} - x_{|y_1,u_\zeta} \right)$$

where $x_{|y_1,u_\zeta}$ denotes the optimal estimate of x using the information available to Player 1, and $x_{|y,u}$ is the optimal estimate of x using the information available Player 2. The matrices K, J, and H are determined from the solutions to Riccati and Sylvester equations. The precise meaning of *optimal estimate* and u_ζ will be explained in Section V. Our results therefore provide a type of separation principle for such problems. Questions of separation are central to decentralized control, and very little is known in the general case. Our results therefore also shed some light on this important issue. Even though the controller is expressed in terms of optimal estimators, these do not all evolve according to a standard Kalman filter (also known as the Kalman–Bucy filter), since Player 1, for example, does not know u_2.

The main contribution of this paper is an explicit state-space formula for the optimal controller, which was not previously known. The realization we find is generically minimal, and computing it is of comparable computational complexity to computing the optimal centralized controller. The solution also gives an intuitive interpretation for the states of the optimal controller.

The paper is organized as follows. In the remainder of the introduction, we give a brief history of decentralized control and the two-player problem in particular. Then, we cover some background mathematics and give a formal statement of the two-player optimal control problem. In Section III, we characterize all structured stabilizing controllers, and show that the two-player control problem can be expressed as an equivalent structured model-matching problem that is convex. In Section IV, we state our main result which contains the optimal controller formulae. We follow up with a discussion of the structure and interpretation of this controller in Section V. The subsequent Section VI gives a detailed proof of the main result. Finally, we conclude in Section VII.

I-A Prior work

If we consider the problem of Section I but remove the structural constraint (1), the problem becomes the well-studied classical $\mathcal{H}_2$ synthesis, solved for example in [34]. The optimal controller is then linear and has as many states as the plant.

The presence of structural constraints greatly complicates the problem, and the resulting decentralized problem has been outstanding since the seminal paper by Witsenhausen [29] in 1968. Witsenhausen posed a related problem for which a nonlinear controller strictly outperforms all linear policies. Not all structural constraints lead to nonlinear optimal controllers, however. For a broad class of decentralized control problems there exists a linear optimal policy [4]. Of recent interest have been classes of problems for which finding the optimal linear controller amounts to solving a convex optimization problem [18, 20, 28]. The two-player problem considered in the present work is one such case.

Despite the benefit of convexity, the search space is infinite-dimensional since we must optimize over transfer functions. Several numerical and analytical approaches for addressing decentralized optimal control exist, including [18, 19, 23, 33]. One particularly relevant numerical approach is to use vectorization, which converts the decentralized problem into an equivalent centralized problem [21]. This conversion process results in a dramatic growth in state dimension, and so the method is computationally intensive and only feasible for small problems. However, it does provide important insight into the problem. Namely, it proves that there exists an optimal linear controller for the two-player problem considered herein that is rational. These results are discussed in Section VI-A.

A drawback of such numerical approaches is that they do not provide an intuitive explanation for what the controller is doing; there is no physical interpretation for the states of the controller. In the centralized case, we have such an interpretation. Specifically, the controller consists of a Kalman filter whose states are estimates of the states of the plant together with a static control gain that corresponds to the solution of an LQR problem. Recent structural results [1, 17] reveal that for general classes of delayed-sharing information patterns, the optimal control policy depends on a summary of the information available and a dynamic programming approach may be used to compute it. There are general results also in the case where not all information is eventually available to all players [30]. However, these dynamic programming results do not appear to easily translate to state-space formulae for linear systems.

For linear systems in which each player eventually has access to all information, explicit formulae were found in [8]. A simple case with varying communication delay was treated in [15]. Cases where plant data are only locally available have also been studied [3, 16].

Certain special cases of the two-player problem have been solved explicitly and clean physical interpretations have been found for the states of the optimal controller. Most notably, the state-feedback case admits an explicit state-space solution using a spectral factorization approach [25]. This approach was also used to address a case with partial output feedback, in which there is output feedback for one player and state feedback for the other [26]. The work of [24] also provided a solution to the state-feedback case using the Möbius transform associated with the underlying poset. Certain special cases were also solved in [6], which gave a method for splitting decentralized optimal control problems into multiple centralized problems. This splitting approach addresses cases other than state-feedback, including partial output-feedback, and dynamically decoupled problems.

In this article, we address the general two-player output-feedback problem. Our approach is perhaps closest technically to the work of [25] using spectral factorization, but uses the factorization to split the problem in a different way, allowing a solution of the general output-feedback problem. We also provide a meaningful interpretation of the states of the optimal controller. This paper is a substantially more general version of the invited paper [12] and the conference paper [13], where the special case of stable systems was considered. We also mention the related work [9] which addresses star-shaped systems in the stable case.

II Preliminaries

We use $\mathbb{Z}_+$ to denote the nonnegative integers. The imaginary unit is i, and we denote the imaginary axis by $\mathrm{i}\mathbb{R}$. A square matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz if all of its eigenvalues have a strictly negative real part. The set $\mathcal{L}_2(\mathrm{i}\mathbb{R})$, or simply $\mathcal{L}_2$, is a Hilbert space of Lebesgue measurable matrix-valued functions $\mathcal{F} : \mathrm{i}\mathbb{R} \to \mathbb{C}^{m\times n}$ with the inner product

$$\langle \mathcal{F}, \mathcal{G} \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{trace}\bigl(\mathcal{F}^*(\mathrm{i}\omega)\mathcal{G}(\mathrm{i}\omega)\bigr)\, \mathrm{d}\omega$$

such that the inner product induced norm $\|\mathcal{F}\|_2 = \langle \mathcal{F}, \mathcal{F} \rangle^{1/2}$ is bounded. We will sometimes write $\mathcal{L}_2^{m\times n}$ to be explicit about the matrix dimensions. As is standard, $\mathcal{H}_2$ is a closed subspace of $\mathcal{L}_2$ with matrix functions analytic in the open right-half plane. $\mathcal{H}_2^\perp$ is the orthogonal complement of $\mathcal{H}_2$ in $\mathcal{L}_2$. We write $\mathcal{R}_p$ to denote the set of proper real rational transfer functions. We also use $\mathcal{R}$ as a prefix to modify other sets to indicate the restriction to real rational functions. So $\mathcal{RL}_2$ is the set of strictly proper rational transfer functions with no poles on the imaginary axis, and $\mathcal{RH}_2$ is the stable subspace of $\mathcal{RL}_2$. The set of stable proper transfer functions is denoted $\mathcal{RH}_\infty$. For the remainder of this paper, whenever we write $\|\mathcal{G}\|_2$, it will always be the case that $\mathcal{G} \in \mathcal{RH}_2$. Every $\mathcal{G} \in \mathcal{R}_p$ has a state-space realization

$$\mathcal{G} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = D + C(sI - A)^{-1}B \qquad \text{with} \qquad \mathcal{G}^* = \left[\begin{array}{c|c} -A^\mathsf{T} & C^\mathsf{T} \\ \hline -B^\mathsf{T} & D^\mathsf{T} \end{array}\right]$$

where $\mathcal{G}^*$ is the conjugate transpose of $\mathcal{G}$. If this realization is chosen to be stabilizable and detectable, then $\mathcal{G} \in \mathcal{RH}_\infty$ if and only if A is Hurwitz, and $\mathcal{G} \in \mathcal{RH}_2$ if and only if A is Hurwitz and $D = 0$. For a thorough introduction to these topics, see [34].

The plant $\mathcal{P} \in \mathcal{R}_p$ maps exogenous inputs w and actuator inputs u to regulated outputs z and measurements y. We seek a control law $u = \mathcal{K}y$ where $\mathcal{K} \in \mathcal{R}_p$ so that the closed-loop map has some desirable properties. The closed-loop map is depicted in Figure 2, and corresponds to the equations

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}, \quad u = \mathcal{K}y \tag{2}$$

The closed-loop map from w to z is given by the lower linear fractional transform (LFT) defined by $F_\ell(\mathcal{P},\mathcal{K}) := \mathcal{P}_{11}+\mathcal{P}_{12}\mathcal{K}\left(I-\mathcal{P}_{22}\mathcal{K}\right)^{-1}\mathcal{P}_{21}$. If $\mathcal{K} = F_\ell(\mathcal{J},\mathcal{Q})$ and $\mathcal{J}$ has a proper inverse, the LFT may be inverted according to $\mathcal{Q} = F_u(\mathcal{J}^{-1},\mathcal{K})$, where F_u denotes the upper LFT, defined as $F_u(\mathcal{M},\mathcal{K}) := \mathcal{M}_{22}+\mathcal{M}_{21}\mathcal{K}\left(I-\mathcal{M}_{11}\mathcal{K}\right)^{-1}\mathcal{M}_{12}$.

We will express the results of this paper in terms of the solutions to algebraic Riccati equations (ARE) and recall here the basic facts. If $D^\mathsf{T}D > 0$, then the following are equivalent.

(i) There exists $X \in \mathbb{R}^{n\times n}$ such that

$$A^\mathsf{T}X + XA + C^\mathsf{T}C - (XB + C^\mathsf{T}D)(D^\mathsf{T}D)^{-1}(B^\mathsf{T}X + D^\mathsf{T}C) = 0$$

and $A - B(D^\mathsf{T}D)^{-1}(B^\mathsf{T}X + D^\mathsf{T}C)$ is Hurwitz

(ii) (A,B) is stabilizable and $\begin{bmatrix} A - \mathrm{i}\omega I & B \\ C & D \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$.

Under these conditions, there exists a unique $X \in \mathbb{R}^{n\times n}$ satisfying (i). This X is symmetric and positive semidefinite, and is called the *stabilizing solution* of the ARE. As a short form we will write $(X,K) = \mathrm{ARE}(A,B,C,D)$ where $K = -(D^\mathsf{T}D)^{-1}(B^\mathsf{T}X + D^\mathsf{T}C)$ is the associated gain.

II-A Block-triangular matrices

Due to the triangular structure of our problem, we also require notation to describe sets of block-lower-triangular matrices. To this end, suppose R is a commutative ring, $m, n \in \mathbb{Z}_+^2$ and $m_i, n_i \geq 0$. We define $\mathrm{lower}(R,m,n)$ to be the set of block-lower-triangular matrices with elements in R partitioned according to the index sets m and n. That is, $X \in \mathrm{lower}(R,m,n)$ if and only if

$$X = \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix} \quad \text{where} \quad X_{ij} \in R^{m_i \times n_j}$$

We sometimes omit the indices and simply write $\mathrm{lower}(R)$. We also define the matrices $E_1 = \begin{bmatrix} I & 0 \end{bmatrix}^\mathsf{T}$ and $E_2 = \begin{bmatrix} 0 & I \end{bmatrix}^\mathsf{T}$, with dimensions to be inferred from context. For example, if we write XE_1 where $X \in \mathrm{lower}(\mathbb{R},m,n)$, then we mean $E_1 \in \mathbb{R}^{(n_1+n_2)\times n_1}$. When writing $A \in \mathrm{lower}(R)$, we allow for the possibility that some of the blocks may be empty. For example, if $m_1 = 0$ then we encounter the trivial case where $\mathrm{lower}(R,m,n) = R^{m_2\times(n_1+n_2)}$.

There is a correspondence between proper transfer functions $\mathcal{G} \in \mathcal{R}_p$ and state-space realizations (A,B,C,D). The next result shows that a specialized form of this correspondence exists when $\mathcal{G} \in \mathrm{lower}(\mathcal{R}_p)$.

Lemma 1. *Suppose $\mathcal{G} \in \mathrm{lower}(\mathcal{R}_p,k,m)$, and a realization for $\mathcal{G}$ is given by (A,B,C,D). Then there exists $n \in \mathbb{Z}_+^2$ and an invertible matrix T such that*

$$TAT^{-1} \in \mathrm{lower}(\mathbb{R},n,n) \qquad TB \in \mathrm{lower}(\mathbb{R},n,m)$$
$$CT^{-1} \in \mathrm{lower}(\mathbb{R},k,n) \qquad D \in \mathrm{lower}(\mathbb{R},k,m)$$

Proof. Partition the state-space matrices according to the partition imposed by k and m.

$$\mathcal{G} = \begin{bmatrix} \mathcal{G}_{11} & 0 \\ \mathcal{G}_{21} & \mathcal{G}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & 0 \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Note that we immediately have $D \in \text{lower}(\mathbb{R}, k, m)$. However, A, B, and C need not have the desired structure. If $k_1 = 0$ or $m_2 = 0$, then $\mathcal{G}_{12}$ is empty, the sparsity pattern is trivial, and any realization (A, B, C, D) will do. Suppose $\mathcal{G}_{12}$ is non-empty and let T be the matrix that transforms $\mathcal{G}_{12}$ into Kalman canonical form. There are typically four blocks in such a decomposition, but since $\mathcal{G}_{12} = 0$, there can be no modes that are both controllable and observable. Apply the same T-transformation to $\mathcal{G}$, and obtain the realization

$$\mathcal{G} = \left[\begin{array}{ccc|cc} A_{\bar{c}o} & 0 & 0 & B_{11} & 0 \\ A_{21} & A_{\bar{c}\bar{o}} & 0 & B_{21} & 0 \\ A_{31} & A_{32} & A_{c\bar{o}} & B_{31} & B_{c\bar{o}} \\ \hline C_{\bar{c}o} & 0 & 0 & D_{11} & 0 \\ C_{21} & C_{22} & C_{23} & D_{21} & D_{22} \end{array}\right] \tag{3}$$

This realization has the desired sparsity pattern, and we notice that there may be many admissible index sets n. For example, the modes $A_{\bar{c}\bar{o}}$ can be included into either the A_{11} block or the A_{22} block. Note that (3) is an admissible realization even if some of the diagonal blocks of A are empty. ■

Lemma 1 also holds for more general sparsity patterns [11].

II-B Problem statement

We seek controllers $\mathcal{K} \in \mathcal{R}_p$ such that when they are connected in feedback to a plant $\mathcal{P} \in \mathcal{R}_p$ as in Figure 2, the plant is stabilized. Consider the interconnection in Figure 3. We say that $\mathcal{K}$ stabilizes $\mathcal{P}$ if the transfer function $(w, u_1, u_2) \to (z, y_1, y_2)$ is well-posed and stable. Well-posedness amounts to $I - \mathcal{P}_{22}(\infty)\mathcal{K}(\infty)$ being nonsingular.

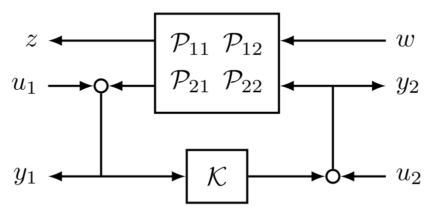

Figure 3: Feedback loop with additional inputs and outputs for analysis of stabilization of two-input two-output systems

The problem addressed in this article may be formally stated as follows. Suppose $\mathcal{P} \in \mathcal{R}_p$ is given. Further suppose that $\mathcal{P}_{22} \in \text{lower}(\mathcal{R}_p, k, m)$. The two-player problem is

$$\begin{array}{ll} \text{minimize} & \left\| \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} \right\|_2 \\ \text{subject to} & \mathcal{K} \in \text{lower}(\mathcal{R}_p, m, k) \\ & \mathcal{K} \text{ stabilizes } \mathcal{P} \end{array} \tag{4}$$

We will also make some additional standard assumptions. We will assume that $\mathcal{P}_{11}$ and $\mathcal{P}_{22}$ are strictly proper, which ensures that the interconnection of Figure 3 is always well-posed, and we will make some technical assumptions on $\mathcal{P}_{12}$ and $\mathcal{P}_{21}$ in order to guarantee the existence and uniqueness of the optimal controller. The first step in our solution to the two-player problem (4) is to deal with the stabilization constraint. This is the topic of the next section.

III Stabilization of triangular systems

Closed-loop stability for decentralized systems is a subtle issue. In the centralized case, the core idea of pole-zero cancellation is ancient, and this was beautifully extended to multivariable system in the Desoer–Chan theory of closed-loop stability [2], where the interconnection of the plant and controller in Figure 3 is considered closed-loop stable if and only if the transfer function $(w, u_1, u_2) \to (z, y_1, y_2)$ is well-posed and stable. Other modeling assumptions are possible, where one either includes different plant uncertainty or different injected and output signals. These assumptions would lead to different definitions of stability. For SISO systems, these two notions were shown to be equivalent, and thus robustness to noise added to communication signals between the plant and the controller is equivalent to this type of robustness to plant modeling error. Several works have proposed extensions of these ideas to decentralized systems, including [27, 31]. However, it is as yet poorly understood exactly what the correspondence is between plant uncertainty and signal uncertainty, and also it remains unclear which the relevant definition of decentralized stability is in practice. We therefore stick to the well-established notion of closed-loop stability used for centralized control systems.

In this section, we provide a state-space characterization of stabilization when both the plant and controller have a block-lower-triangular structure. Specifically, we give necessary and sufficient conditions under which a block-lower-triangular stabilizing controller exists, and we provide a parameterization of all such controllers akin to the Youla parameterization [32]. Many of the results in this section appeared in [11] and similar results also appeared in [18]. Throughout this section, we assume that the plant $\mathcal{P}$ satisfies

$$\begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \qquad \text{is a minimal realization} \tag{5}$$

We further assume that the $\mathcal{P}_{22}$ subsystem is block-lower-triangular. Since we have $\mathcal{P}_{22} \in \operatorname{lower}(\mathcal{R}_p)$, Lemma 1 allows us to assume without loss of generality that the matrices A, B_2, and C_2 have the form

$$A := \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \qquad B_2 := \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \qquad C_2 := \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix} \tag{6}$$

where $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $B_{ij} \in \mathbb{R}^{n_i \times m_j}$ and $C_{ij} \in \mathbb{R}^{k_i \times n_j}$. The following result gives necessary and sufficient conditions under which a there exists a structured stabilizing controller.

Lemma 2. *Suppose $\mathcal{P} \in \mathcal{R}_p$ and $\mathcal{P}_{22} \in \operatorname{lower}(\mathcal{R}_p, k, m)$. Let (A, B, C, D) be a minimal realization of $\mathcal{P}$ that satisfies (5)–(6). There exists $\mathcal{K}_0 \in \operatorname{lower}(\mathcal{R}_p, m, k)$ such that $\mathcal{K}_0$ stabilizes $\mathcal{P}$ if and only if both*

(i) (C_{11}, A_{11}, B_{11}) is stabilizable and detectable, and

(ii) (C_{22}, A_{22}, B_{22}) is stabilizable and detectable.

In this case, one such controller is

$$\mathcal{K}_0 = \left[\begin{array}{c|c} A + B_2 K_d + L_d C_2 & -L_d \\ \hline K_d & 0 \end{array}\right] \tag{7}$$

$$\text{where} \quad K_d := \begin{bmatrix} K_1 & \\ & K_2 \end{bmatrix} \quad \text{and} \quad L_d := \begin{bmatrix} L_1 & \\ & L_2 \end{bmatrix}$$

and K_i and L_i are chosen such that $A_{ii} + B_{ii} K_i$ and $A_{ii} + L_i C_{ii}$ are Hurwitz for $i = 1, 2$.

Proof. ($\Longleftarrow$) Suppose that (i) and (ii) hold. Note that $A + B_2K_d$ and $A + L_dC_2$ are Hurwitz by construction, thus (C_2, A, B_2) is stabilizable and detectable, and it is thus immediate that (7) is stabilizing. Due to the block-diagonal structure of K_d and L_d, it is straightforward to verify that $\mathcal{K}_0 \in \text{lower}(\mathcal{R}_p)$.

($\Longrightarrow$) Suppose $\mathcal{K}_0 \in \text{lower}(\mathcal{R}_p, m, k)$ stabilizes $\mathcal{P}$. Because $\mathcal{P}$ is minimal, we must have that the realization (C_2, A, B_2) is stabilizable and detectable. By Lemma 1, we may assume that $\mathcal{K}_0$ has a minimal realization with $A_K, B_K, C_K, D_K \in \text{lower}(\mathbb{R})$. The closed-loop generator $\bar{A}$ is Hurwitz, where

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix}$$

It follows that

$$\left(\begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix}, \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \right) \text{ is stabilizable}$$

and hence by the PBH test, (A, B_2) and (A_K, B_K) are stabilizable and similarly (C_2, A) and (C_K, A_K) are detectable.

Each block of $\bar{A}$ is block-lower-triangular. Viewing $\bar{A}$ as a block 4×4 matrix, transform $\bar{A}$ using a matrix T that permutes states 2 and 3. Now $T^{-1}\bar{A}T$ is Hurwitz implies that the two 2×2 blocks on the diagonal are Hurwitz. But these diagonal blocks are precisely the closed-loop $\bar{A}$ matrices corresponding to the 11 and 22 subsystems. Applying the PBH argument to the diagonal blocks of $T^{-1}\bar{A}T$ implies that (C_{ii}, A_{ii}, B_{ii}) is stabilizable and detectable for $i = 1, 2$ as desired. ∎

Note that the centralized characterization of stabilization, as in [34, Lemma 12.1], only requires that (C_2, A, B_2) be stabilizable and detectable. The conditions in Lemma 2 are stronger because of the additional structural constraint.

We may also characterize the stabilizability of just the 22 block, which we state as a corollary.

Corollary 3. *Suppose $\mathcal{P}_{22} \in \text{lower}(\mathcal{R}_p, k, m)$, and let (A, B_2, C_2, D_{22}) be a minimal realization of $\mathcal{P}_{22}$ that satisfies (6). There exists $\mathcal{K}_0 \in \text{lower}(\mathcal{R}_p, m, k)$ such that $\mathcal{K}_0$ stabilizes $\mathcal{P}_{22}$ if and only if both*

(i) (C_{11}, A_{11}, B_{11}) is stabilizable and detectable, and

(ii) (C_{22}, A_{22}, B_{22}) is stabilizable and detectable.

Indeed, there exist block-lower transfer matrices that cannot be stabilized by a block-lower controller. For example,

$$\mathcal{P}_{22} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s-1} & \frac{1}{s+1} \end{bmatrix} = \left[\begin{array}{ccc|cc} -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array} \right]$$

The above realization is minimal, but the grouping of states into blocks is not unique. We may group the unstable mode either in the A_{11} block or in the A_{22} block, which corresponds to $n = (2, 1)$ or $n = (1, 2)$, respectively. The former leads to an undetectable (C_{11}, A_{11}) while the latter leads to an unstabilizable (A_{22}, B_{22}). By Corollary 3, this plant cannot be stabilized by a block-lower-triangular controller. However, centralized stabilizing controllers exist due to the minimality of the realization.

Note that a stabilizable $\mathcal{P}_{22}$ may have an off-diagonal block that is unstable. An example is:

$$\mathcal{P}_{22} = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{s-1} & \frac{1}{s+1} \end{bmatrix} \qquad \mathcal{K}_0 = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

We now treat the parameterization of all stabilizing controllers. The following result was proved in [18].

Theorem 4. *Suppose the conditions of Lemma 2 hold, and (C_{11}, A_{11}, B_{11}) and (C_{22}, A_{22}, B_{22}) are both stabilizable and detectable. Define K_d and L_d as in Lemma 2. The set of all $\mathcal{K} \in \text{lower}(\mathcal{R}_p, m, k)$ that stabilize $\mathcal{P}$ is*

$$\left\{ \mathcal{F}_\ell(\mathcal{J}_d, \mathcal{Q}) \,\middle|\, \mathcal{Q} \in \text{lower}(\mathcal{RH}_\infty, m, k) \right\}$$

$$\textit{where} \quad \mathcal{J}_d = \left[\begin{array}{c|cc} A + B_2K_d + L_dC_2 & -L_d & B_2 \\ \hline K_d & 0 & I \\ -C_2 & I & 0 \end{array}\right] \tag{8}$$

Proof. If we relax the constraint that $\mathcal{Q}$ be lower triangular, then this is the standard parameterization of all centralized stabilizing controllers [34, Theorem 12.8]. It suffices to show that the map from $\mathcal{Q}$ to $\mathcal{K}$ and its inverse are structure-preserving. Since each block of the state-space realization of $\mathcal{J}_d$ is in lower($\mathbb{R}$), we have $(\mathcal{J}_d)_{ij} \in \text{lower}(\mathcal{R}_p)$. Thus $F_\ell(\mathcal{J}_d, \cdot)$ preserves lower triangularity on its domain. This also holds for the inverse map $F_u(\mathcal{J}_d^{-1}, \cdot)$, since

$$\mathcal{J}_d^{-1} = \left[\begin{array}{c|cc} A & B_2 & -L_d \\ \hline C_2 & 0 & I \\ -K_d & I & 0 \end{array}\right]$$

■

As in the centralized case, this parameterization of stabilizing controllers allows us to rewrite the closed-loop map in terms of $\mathcal{Q}$. After some simplification, we obtain $F_\ell(\mathcal{P}, \mathcal{K}) = \mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}\mathcal{T}_{21}$ where $\mathcal{T}$ has the realization

$$\begin{bmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} A_{Kd} & -B_2K_d & B_1 & B_2 \\ 0 & A_{Ld} & B_{Ld} & 0 \\ \hline C_{Kd} & -D_{12}K_d & 0 & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right] \tag{9}$$

and we have used the following shorthand notation.

$$\begin{aligned} A_{Kd} &:= A + B_2K_d & A_{Ld} &:= A + L_dC_2 \\ C_{Kd} &:= C_1 + D_{12}K_d & B_{Ld} &:= B_1 + L_dD_{21} \end{aligned} \tag{10}$$

Combining the results above gives the following important equivalence between the two-player output-feedback problem (4) and a structured model-matching problem.

Corollary 5. *Suppose the conditions of Theorem 4 hold. Then $\mathcal{Q}_{\text{opt}}$ is a minimizer for*

$$\begin{aligned} &\text{minimize} && \left\| \mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}\mathcal{T}_{21} \right\|_2 \\ &\text{subject to} && \mathcal{Q} \in \text{lower}(\mathcal{RH}_\infty) \end{aligned} \tag{11}$$

if and only if $\mathcal{K}_{\text{opt}} = F_\ell(\mathcal{J}_d, \mathcal{Q}_{\text{opt}})$ is a minimizer for the the two-player output-feedback problem (4)*. Here $\mathcal{J}_d$ is given by* (8)*, and $\mathcal{T}$ is defined in* (9)*. Furthermore, $\mathcal{Q}_{\text{opt}}$ is unique if and only if $\mathcal{K}_{\text{opt}}$ is unique.*

Corollary 5 gives an equivalence between the two-player output-feedback problem (4) and the two-player stable model-matching problem (11). The new formulation should be easier to solve than the output-feedback version because it is convex and the $\mathcal{T}_{ij}$ are stable. However, its solution is still not straightforward, because the problem remains infinite-dimensional and there is a structural constraint on $\mathcal{Q}$.

IV Main result

In this section, we present our main result: a solution to the two-player output-feedback problem. First, we state our assumptions and list the equations that must be solved. We assume the plant satisfies (5) and the matrices A, B_2, C_2 have the form (6). We further assume that A_{11} and A_{22} have non-empty dimensions, so $n_i \neq 0$. This avoids trivial special cases and allows us to streamline the results. To ease notation, we define the following cost and covariance matrices

$$\begin{aligned} \begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} &:= \begin{bmatrix} C_1^\mathsf{T} C_1 & C_1^\mathsf{T} D_{12} \\ D_{12}^\mathsf{T} C_1 & D_{12}^\mathsf{T} D_{12} \end{bmatrix} = \begin{bmatrix} C_1 & D_{12} \end{bmatrix}^\mathsf{T} \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \\ \begin{bmatrix} W & U^\mathsf{T} \\ U & V \end{bmatrix} &:= \begin{bmatrix} B_1 B_1^\mathsf{T} & B_1 D_{21}^\mathsf{T} \\ D_{21} B_1^\mathsf{T} & D_{21} D_{21}^\mathsf{T} \end{bmatrix} = \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix}^\mathsf{T} \end{aligned} \tag{12}$$

Our main assumptions are as follows.

A1) $D_{12}^\mathsf{T} D_{12} > 0$

A2) (A_{11}, B_{11}) and (A_{22}, B_{22}) are stabilizable

A3) $\begin{bmatrix} A - \mathrm{i}\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$

A4) $D_{21} D_{21}^\mathsf{T} > 0$

A5) (C_{11}, A_{11}) and (C_{22}, A_{22}) are detectable

A6) $\begin{bmatrix} A - \mathrm{i}\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all $\omega \in \mathbb{R}$

We will also require the solutions to four AREs

$$\begin{aligned} (X, K) &= \mathrm{ARE}(A, B_2, C_1, D_{12}) & (Y, L^\mathsf{T}) &= \mathrm{ARE}(A^\mathsf{T}, C_2^\mathsf{T}, B_1^\mathsf{T}, D_{21}^\mathsf{T}) \\ (\tilde{X}, J) &= \mathrm{ARE}(A_{22}, B_{22}, C_1 E_2, D_{12} E_2) & (\tilde{Y}, M^\mathsf{T}) &= \mathrm{ARE}(A_{11}^\mathsf{T}, C_{11}^\mathsf{T}, E_1^\mathsf{T} B_1^\mathsf{T}, E_1^\mathsf{T} D_{21}^\mathsf{T}) \end{aligned} \tag{13}$$

Finally, we must solve the following simultaneous linear equations for $\Phi, \Psi \in \mathbb{R}^{n_2 \times n_1}$

$$\begin{aligned} (A_{22} + B_{22} J)^\mathsf{T} \Phi + \Phi (A_{11} + M C_{11}) - (\tilde{X} - X_{22})(\Psi C_{11}^\mathsf{T} + U_{12}^\mathsf{T}) V_{11}^{-1} C_{11} \\ + \left(\tilde{X} A_{21} + J^\mathsf{T} S_{12}^\mathsf{T} + Q_{21} - X_{21} M C_{11} \right) = 0 \end{aligned} \tag{14}$$

$$\begin{aligned} (A_{22} + B_{22} J) \Psi + \Psi (A_{11} + M C_{11})^\mathsf{T} - B_{22} R_{22}^{-1} (B_{22}^\mathsf{T} \Phi + S_{12}^\mathsf{T})(\tilde{Y} - Y_{11}) \\ + \left(A_{21} \tilde{Y} + U_{12}^\mathsf{T} M^\mathsf{T} + W_{21} - B_{22} J Y_{21} \right) = 0 \end{aligned} \tag{15}$$

and define the associated gains $\hat{K} \in \mathbb{R}^{(m_1+m_2) \times (n_1+n_2)}$ and $\hat{L} \in \mathbb{R}^{(n_1+n_2) \times (k_1+k_2)}$

$$\hat{K} := \begin{bmatrix} 0 & 0 \\ -R_{22}^{-1} \left(B_{22}^\mathsf{T} \Phi + S_{12}^\mathsf{T} \right) & J \end{bmatrix} \qquad \hat{L} := \begin{bmatrix} M & 0 \\ -(\Psi C_{11}^\mathsf{T} + U_{12}^\mathsf{T}) V_{11}^{-1} & 0 \end{bmatrix} \tag{16}$$

For convenience, we define the Hurwitz matrices

$$\begin{aligned} A_K &:= A + B_2 K & A_L &:= A + L C_2 \\ A_J &:= A_{22} + B_{22} J & A_M &:= A_{11} + M C_{11} \\ \hat{A} &:= A + B_2 \hat{K} + \hat{L} C_2 \end{aligned} \tag{17}$$

Note that A_K, A_L, A_J, A_M are all Hurwitz by construction, and $\hat{A}$ is Hurwitz as well, because it is block-lower-triangular and its block-diagonal entries are A_M and A_J. The matrices Φ and Ψ have physical interpretations, as do the gains $\hat{K}$ and $\hat{L}$. These will be explained in Section V. The main result of this paper is Theorem 6, given below.

Theorem 6. *Suppose $\mathcal{P} \in \mathcal{R}_p$ satisfies* (5)–(6) *and Assumptions A1–A6. Consider the two-player output feedback problem as stated in* (4)

$$\begin{aligned} \text{minimize} \quad & \left\| \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} \right\|_2 \\ \text{subject to} \quad & \mathcal{K} \in \text{lower}(\mathcal{R}_p, m, k) \\ & \mathcal{K} \text{ stabilizes } \mathcal{P} \end{aligned}$$

(i) There exists a unique optimal $\mathcal{K}$.

(ii) Suppose Φ, Ψ satisfy the linear equations (14)–(15). *Then the optimal controller is*

$$\mathcal{K}_{\text{opt}} = \left[\begin{array}{cc|c} A + B_2K + \hat{L}C_2 & 0 & -\hat{L} \\ B_2K - B_2\hat{K} & A + LC_2 + B_2\hat{K} & -L \\ \hline K - \hat{K} & \hat{K} & 0 \end{array}\right] \tag{18}$$

where K, L, $\hat{K}$, $\hat{L}$ are defined in (13) *and* (16).

(iii) There exist Φ, Ψ satisfying (14)–(15).

Proof. A complete proof is provided in Section VI. ■

An alternative realization for the optimal controller is

$$\mathcal{K}_{\text{opt}} = \left[\begin{array}{cc|c} A + B_2K + \hat{L}C_2 & 0 & \hat{L} \\ LC_2 - \hat{L}C_2 & A + LC_2 + B_2\hat{K} & L - \hat{L} \\ \hline -K & -\hat{K} & 0 \end{array}\right] \tag{19}$$

V Structure of the optimal controller

In this section, we will discuss several features and consequences of the optimal controller exhibited in Theorem 6. First, a brief discussion on duality and symmetry. We then show a structural result, that the states of the optimal controller have a natural stochastic interpretation as minimum-mean-square-error estimates. This will lead to a useful interpretation of the matrices Φ and Ψ defined in (14)–(15). We then compute the $\mathcal{H}_2$-norm of the optimally controlled system, and characterize the portion of the cost attributed to the decentralization constraint. Finally, we show how our main result specializes to many previously solved cases appearing in the literature.

V-A Symmetry and duality

The solution to the two-player output-feedback problem has nice symmetry properties that are perhaps unexpected given the network topology. Player 2 sees more information than Player 1, so one might expect Player 2's optimal policy to be more complicated than that of Player 1. Yet, this is not the case. Player 2 observes all the measurements but only controls u_2, so only influences subsystem 1. In contrast, Player 1 only observes subsystem 1, but controls u_1, which in turn influences both subsystems. This duality is reflected in (13)–(16).

If we transpose the system variables and swap Player 1 and Player 2, then every quantity related to control of the original system becomes a corresponding quantity related to estimation of the transformed system. More formally, if we define

$$A^{\ddagger} = \begin{bmatrix} A_{22}^{\mathsf{T}} & A_{12}^{\mathsf{T}} \\ A_{21}^{\mathsf{T}} & A_{11}^{\mathsf{T}} \end{bmatrix}$$

then the transformation $(A, B_2, C_2) \mapsto (A^{\ddagger}, C_2^{\ddagger}, B_2^{\ddagger})$ leads to

$$(X, K) \mapsto (Y^{\ddagger}, L^{\ddagger}), \;\; (\hat{X}, \hat{K}) \mapsto (\hat{Y}^{\ddagger}, \hat{L}^{\ddagger}), \;\text{ and }\; \hat{A} \mapsto \hat{A}^{\ddagger}.$$

This is analogous to the well-known correspondence between Kalman filtering and optimal control in the centralized case.

V-B Gramian equations

In this subsection, we derive useful algebraic relations that follow from Theorem 6 and provide a stochastic interpretation. Recall the plant equations, written in differential form.

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ y &= C_2 x + D_{21} w \end{aligned} \tag{20}$$

The optimal controller (18) is given in Theorem 6. Label its states as ζ and ξ, and place each equation into the following canonical observer form.

$$\dot{\zeta} = A\zeta + B_2 \hat{u} - \hat{L}(y - C_2\zeta) \tag{21}$$

$$\hat{u} = K\zeta \tag{22}$$

$$\dot{\xi} = A\xi + B_2 u - L(y - C_2\xi) \tag{23}$$

$$u = K\zeta + \hat{K}(\xi - \zeta) \tag{24}$$

We will see later that this choice of coordinates leads to a natural interpretation in terms of steady-state Kalman filters. Our first result is a Lyapunov equation unique to the two-player problem.

Lemma 7. *Suppose Φ, Ψ is a solution of* (14)–(15)*. Define $\hat{L}$ and $\hat{A}$ according to* (16)–(17)*. There exists a unique matrix $\hat{Y}$ satisfying the equation*

$$\hat{A}(\hat{Y} - Y) + (\hat{Y} - Y)\hat{A}^{\mathsf{T}} + (\hat{L} - L)V(\hat{L} - L)^{\mathsf{T}} = 0 \tag{25}$$

Further, $\hat{Y} \geq Y$, $\hat{Y}_{11} = \tilde{Y}$, $\hat{Y}_{21} = \Psi$, and

$$\hat{L} = -\big(\hat{Y} C_2^{\mathsf{T}} + U^{\mathsf{T}}\big) E_1 V_{11}^{-1} E_1^{\mathsf{T}} \tag{26}$$

Proof. Since $\hat{A}$ is stable and $(\hat{L} - L)V(\hat{L} - L)^{\mathsf{T}} \geq 0$, it follows from standard properties of Lyapunov equations that (25) has a unique solution and it satisfies $\hat{Y} - Y \geq 0$. Right-multiplying (25) by E_1 gives

$$\hat{A}(\hat{Y}E_1 - YE_1) + (\hat{Y}E_1 - YE_1)A_M^{\mathsf{T}} + (\hat{L} - L)V(E_1 M^{\mathsf{T}} - L^{\mathsf{T}} E_1) = 0$$

This equation splits into two Sylvester equations, the first in $\hat{Y}_{11}$, and the second in $\hat{Y}_{21}$. Both have unique solutions. Upon comparison with (13), (15), (16), one finds that $\hat{Y}_{11} = \tilde{Y}$ and $\hat{Y}_{21} = \Psi$. Note that, since Φ is fixed, Ψ is uniquely determined. Similarly comparing with (16) verifies (26). ∎

Our next observation is that under the right coordinates, the controllability Gramian of the closed-loop map is block-diagonal.

Theorem 8. *Suppose we have the plant described by* (20) *and the optimal controller given by* (21)–(24)*. The closed-loop map for one particular choice of coordinates is*

$$\begin{bmatrix} \dot{\zeta} \\ \dot{\xi} - \dot{\zeta} \\ \dot{x} - \dot{\xi} \end{bmatrix} = \begin{bmatrix} A_K & -\hat{L}C_2 & -\hat{L}C_2 \\ 0 & \hat{A} & (\hat{L}-L)C_2 \\ 0 & 0 & A_L \end{bmatrix} \begin{bmatrix} \zeta \\ \xi - \zeta \\ x - \xi \end{bmatrix} + \begin{bmatrix} -\hat{L}D_{21} \\ (\hat{L}-L)D_{21} \\ B_1 + LD_{21} \end{bmatrix} w$$

which we write compactly as $\dot{q} = A_c q + B_c w$*. Let* Θ *be the controllability Gramian, i.e. the solution* $\Theta \geq 0$ *to the Lyapunov equation* $A_c\Theta + \Theta A_c^{\mathsf{T}} + B_c B_c^{\mathsf{T}} = 0$*. Then*

$$\Theta = \begin{bmatrix} Z & 0 & 0 \\ 0 & \hat{Y} - Y & 0 \\ 0 & 0 & Y \end{bmatrix}$$

where Z *satisfies the Lyapunov equation*

$$A_K Z + Z A_K^{\mathsf{T}} + \hat{L} V \hat{L}^{\mathsf{T}} = 0$$

Proof. Uniqueness of Θ follows because A_c is Hurwitz, and $\Theta \geq 0$ follows because A_c is Hurwitz and $B_c B_c^{\mathsf{T}} \geq 0$. The solution can be verified by direct substitution and by making use of the identities in Lemma 7. ∎

The observation in Theorem 8 lends itself to a statistical interpretation. If w is white Gaussian noise with unit intensity, the steady-state distribution of the random vector q will have covariance Θ. Since Θ is block-diagonal, the steady-state distributions of the components ζ, $\xi-\zeta$, and $x-\xi$ are mutually independent. As expected from our discussion on duality, the algebraic relations derived in Lemma 7 may be dualized to obtain a new result, which we now state without proof.

Lemma 9. *Suppose* Φ, Ψ *is a solution of* (14)–(15)*. Define* $\hat{K}$ *and* $\hat{A}$ *according to* (16)–(17)*. There exists a unique matrix* $\hat{X}$ *satisfying the equation*

$$\hat{A}^{\mathsf{T}}(\hat{X} - X) + (\hat{X} - X)\hat{A} + (\hat{K} - K)^{\mathsf{T}} R (\hat{K} - K) = 0 \tag{27}$$

Further, $\hat{X} \geq X$*,* $\hat{X}_{22} = \tilde{X}$*,* $\hat{X}_{21} = \Phi$*, and*

$$\hat{K} = -E_2 R_{22}^{-1} E_2^{\mathsf{T}} \left(B_2^{\mathsf{T}} \hat{X} + S^{\mathsf{T}} \right) \tag{28}$$

The dual of Theorem 8 can be obtained by expressing the closed-loop map in the coordinates $(x, x-\zeta, x-\xi)$ instead. In these coordinates, one can show that the observability Gramian is also block-diagonal.

V-C Estimation structure

In this subsection, we show that the states ζ and ξ of optimal controller (21)–(24) may be interpreted as suitably defined Kalman filters. Given the dynamics (20), the estimation map and corresponding Kalman filter $\hat{x}$ is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline I & 0 & 0 \\ C_2 & D_{21} & 0 \end{array}\right] \begin{bmatrix} w \\ u \end{bmatrix}, \qquad \hat{x} = \left[\begin{array}{c|cc} A + LC_2 & -L & B_2 \\ \hline I & 0 & 0 \end{array}\right] \begin{bmatrix} y \\ u \end{bmatrix} \tag{29}$$

where L and A_L are defined in (13) and (17) respectively. We will show that this Kalman filter is the result of an $\mathcal{H}_2$ optimization problem and we will define it as such. This operator-based definition will allow us to make precise claims without the need for stochastics. Before stating our definition, we illustrate an important subtlety regarding the input u; in order to make the notion of a Kalman filter unambiguous, one must be careful to specify which signals will be treated as inputs. Consider the following numerical instance of (29).

$$\begin{bmatrix} x \\ y \end{bmatrix} = \left[\begin{array}{c|cc|c} -4 & 3 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} w \\ u \end{bmatrix}, \qquad \hat{x} = \left[\begin{array}{c|cc} -5 & 1 & 1 \\ \hline 1 & 0 & 0 \end{array}\right] \begin{bmatrix} y \\ u \end{bmatrix} \tag{30}$$

Suppose we know the control law u. We may then eliminate u from the estimator (30). For example,

$$u = \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & 0 \end{array}\right] y \quad \text{leads to} \quad \hat{x} = \left[\begin{array}{cc|c} -5 & 1 & 1 \\ 0 & 1 & 1 \\ \hline 1 & 0 & 0 \end{array}\right] y \tag{31}$$

The new estimator dynamics are unstable because we chose an unstable control law. Consider estimating x again, but this time using the control law a priori. Eliminating u from the plant first and then computing the Kalman filter leads to

$$\begin{bmatrix} x \\ y \end{bmatrix} = \left[\begin{array}{cc|cc} -4 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] w, \quad \hat{x} = \left[\begin{array}{cc|c} -7 & 1 & 3 \\ -12 & 1 & 13 \\ \hline 1 & 0 & 0 \end{array}\right] y \tag{32}$$

The estimators (31) and (32) are different. Indeed,

$$\hat{x} = \left(\frac{s}{s^2 + 4s - 5}\right) y \quad \text{and} \quad \hat{x} = \left(\frac{3s + 10}{s^2 + 6s + 5}\right) y$$

in (31) and (32) respectively. In general, open-loop and closed-loop estimation are different so any sensible definition of Kalman filtering must account for this by specifying which inputs (if any) will be estimated in open-loop. For centralized optimal control, the correct filter to use is given by (29)–(30).

Our definition of an *estimator* is as follows. Note that similar optimization-based definitions have appeared in the literature, as in [22].

Definition 10. *Suppose* $\mathcal{G} \in \mathcal{R}_p^{(p_1+p_2)\times(q_1+q_2)}$ *is given by*

$$\mathcal{G} = \begin{bmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} \\ \mathcal{G}_{21} & \mathcal{G}_{22} \end{bmatrix}$$

and $\mathcal{G}_{21}(\mathrm{i}\omega)$ *has full row rank for all* $\omega \in \mathbb{R} \cup \{\infty\}$. *Define the estimator of* $\mathcal{G}$ *to be* $\mathcal{G}^{\text{est}} \in \mathcal{R}_p^{p_1 \times (p_2+q_1)}$ *partitioned according to* $\mathcal{G}^{\text{est}} = \begin{bmatrix} \mathcal{G}_1^{\text{est}} & \mathcal{G}_2^{\text{est}} \end{bmatrix}$ *where*

$$\begin{aligned} \mathcal{G}_1^{\text{est}} &= \mathop{\arg\min}_{\substack{\mathcal{F} \in \mathcal{RH}_2 \\ \mathcal{G}_{11} - \mathcal{F}\mathcal{G}_{21} \in \mathcal{RH}_2}} \left\| \mathcal{G}_{11} - \mathcal{F}\mathcal{G}_{21} \right\|_2 \\ \mathcal{G}_2^{\text{est}} &= \mathcal{G}_{12} - \mathcal{G}_1^{\text{est}} \mathcal{G}_{22} \end{aligned}$$

Note that under the assumptions of Definition 10, the estimator $\mathcal{G}^{\text{est}}$ is unique. We will show existence of $\mathcal{G}^{\text{est}}$ for particular $\mathcal{G}$ below. We now define the following notation.

Definition 11. *Suppose $\mathcal{G}$ and $\mathcal{G}^{\text{est}}$ are as in Definition 10. If*

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathcal{G} \begin{bmatrix} w \\ u \end{bmatrix}$$

then we use the notation $x_{|y,u}$ to mean

$$x_{|y,u} := \mathcal{G}^{\text{est}} \begin{bmatrix} y \\ u \end{bmatrix}$$

This notation is motivated by the property that

$$x - x_{|y,u} = (\mathcal{G}_{11} - \mathcal{G}_1^{\text{est}} \mathcal{G}_{21}) w$$

The stochastic interpretation of Definition 10 is therefore that $\mathcal{G}_1^{\text{est}}$ is chosen to minimize the mean square estimation error assuming w is white Gaussian noise with unit intensity. In the following lemma, we show that the quantity $x_{|y,u}$ as defined in Definitions 10 and 11 is the usual steady-state Kalman filter for estimating the state x using the measurements y and inputs u, as given in (29).

Lemma 12. *Let $\mathcal{G}$ be*

$$\mathcal{G} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline I & 0 & 0 \\ C_2 & D_{21} & 0 \end{array}\right]$$

and suppose Assumptions A4–A6 hold. Then

$$\mathcal{G}^{\text{est}} = \left[\begin{array}{c|cc} A_L & -L & B_2 \\ \hline I & 0 & 0 \end{array}\right]$$

where L and A_L are given by (13) *and* (17) *respectively.*

Proof. A proof of a more general result that includes preview appears in [7, Theorem 4.8]. Roughly, one begins by applying the change of variables $\mathcal{F} = \mathcal{G}_1^{\text{est}} + \bar{\mathcal{F}}$, and then parameterizing all admissible $\bar{\mathcal{F}}$ using a coprime factorization of $\mathcal{G}_{21}$. This yields the following equivalent unconstrained problem

$$\begin{array}{ll} \text{minimize} & \left\| \left[\begin{array}{c|c} A_L & B_1 + LD_{21} \\ \hline I & 0 \end{array}\right] - \mathcal{Q} \left[\begin{array}{c|c} A_L & B_1 + LD_{21} \\ \hline C_2 & D_{21} \end{array}\right] \right\|_2 \\ \text{subject to} & \mathcal{Q} \in \mathcal{RH}_2 \end{array}$$

One may check that $\mathcal{Q} = 0$ is optimal for the unconstrained problem, and hence $\bar{\mathcal{F}} = 0$ as required, by verifying the following orthogonality condition.

$$\left[\begin{array}{c|c} A_L & B_1 + LD_{21} \\ \hline I & 0 \end{array}\right] \left[\begin{array}{c|c} A_L & B_1 + LD_{21} \\ \hline C_2 & D_{21} \end{array}\right]^* \in \mathcal{H}_2^\perp \tag{33}$$

To see why (33) holds, multiply the realizations together and use the state transformation $(x_1, x_2) \mapsto (x_1 - Yx_2, x_2)$, where Y is given in (13). See [34, Lemma 14.3] for an example of such a computation. ∎

Comparing the result of Lemma 12 with the ξ-state of the optimal two-player controller (23), we notice the following consequence.

Corollary 13. *Suppose the conditions of Theorem 6 are satisfied, and the controller states are labeled as in* (21)–(24)*. Then* $x_{|y,u} = \xi$*, where the estimator is defined by the map* $\mathcal{G} : (w, u) \to (x, y)$ *induced by the plant* (20)*.*

The above result means that ξ, one of the states of the optimal controller, is the usual Kalman filter estimate of x given y and u. The next result is more difficult, and gives the analogous result for the other state. The next theorem is our main structural result. We show that ζ may also be interpreted as an optimal estimator in the sense of Definitions 10 and 11.

Theorem 14. *Suppose the conditions of Theorem 6 are satisfied, and label the states of the controller as in* (21)–(24)*. Define also*

$$u_\zeta := (K - \hat{K})\zeta, \qquad u_\xi := \hat{K}\xi \tag{34}$$

so that $u = u_\zeta + u_\xi$*. Then*

$$\begin{bmatrix} x \\ \xi \\ u \end{bmatrix}_{|y_1, u_\zeta} = \begin{bmatrix} \zeta \\ \zeta \\ \hat{u} \end{bmatrix}$$

The estimator is defined by the map $\mathcal{G} : (w, u_\zeta) \to (x, \xi, u, y_1)$ *induced by* (34)*, the plant* (20)*, and the controller* (21)–(24)*.*

Proof. The proof parallels that of Lemma 12, so we omit most of the details. Straightforward algebra gives

$$\mathcal{G} = \left[\begin{array}{cc|cc} A & B_2\hat{K} & B_1 & B_2 \\ -LC_2 & A + LC_2 + B_2\hat{K} & -LD_{21} & B_2 \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & \hat{K} & 0 & I \\ E_1^{\mathsf{T}} C_2 & 0 & E_1^{\mathsf{T}} D_{21} & 0 \end{array}\right]$$

After substituting $\mathcal{F} = \mathcal{G}_1^{\text{est}} + \bar{\mathcal{F}}$, computing the coprime factorization, and changing to more convenient coordinates, the resulting unconstrained optimization problem is

$$\begin{aligned} \text{minimize} \quad & \left\| \left[\begin{array}{cc|c} \hat{A} & (\hat{L} - L)C_2 & (\hat{L} - L)D_{21} \\ 0 & A_L & B_2 + LD_{21} \\ \hline I & I & 0 \\ I & 0 & 0 \\ K & 0 & 0 \end{array}\right] - \mathcal{Q}\mathcal{D} \right\|_2 \\ \text{subject to} \quad & \mathcal{Q} \in \mathcal{RH}_2 \end{aligned}$$

where $\hat{A}$ is defined in (17), and

$$\mathcal{D} = \left[\begin{array}{cc|c} \hat{A} & (\hat{L} - L)C_2 & (\hat{L} - L)D_{211} \\ 0 & A_L & B_2 + LD_{21} \\ \hline E_1^{\mathsf{T}} C_2 & E_1^{\mathsf{T}} C_2 & E_1^{\mathsf{T}} D_{21} \end{array}\right]$$

Optimality of $\mathcal{Q} = 0$ is established by verifying the analogous orthogonality relationship to (33). This is easily done using the Gramian identity from Theorem 8. Then we have

$$\mathcal{G}^{\text{est}} = \left[\begin{array}{c|cc} \hat{A} & -\hat{L}E_1 & B_2 \\ \hline I & 0 & 0 \\ I & 0 & 0 \\ K & 0 & 0 \end{array}\right]$$

Comparing with (21), we can see that

$$\begin{bmatrix} \zeta \\ \zeta \\ \hat{u} \end{bmatrix} = \mathcal{G}^{\text{est}} \begin{bmatrix} y_1 \\ u_\zeta \end{bmatrix}$$

and the result follows. ■

The result of Theorem 14 that $\zeta = x_{|y_1,u_\zeta}$ has a clear interpretation that ζ is a Kalman filter of x. However, this result assumes that Player 2 is implementing the optimal policy for u_2. This is important because the optimal estimator gain depends explicitly on this choice of policy. In contrast, in the centralized case, the optimal estimation gain does not depend on the choice of control policy.

Note that using u_1 or $\hat{u}$ instead of u_ζ in Theorem 14 yields an incorrect interpretation of ζ. In these coordinates,

$$\zeta = \left[\begin{array}{c|cc} A + B_2E_2E_2^\mathsf{T}K + \hat{L}C_2 & -\hat{L}E_1 & B_2E_1 \\ \hline I & 0 & 0 \end{array}\right] \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} \quad \text{or}$$

$$\zeta = \left[\begin{array}{c|cc} A + B_2K + \hat{L}C_2 & -\hat{L}E_1 & B_2 \\ \hline I & 0 & 0 \end{array}\right] \begin{bmatrix} y_1 \\ \hat{u} \end{bmatrix}$$

In general, neither of these maps is stable, so for these choices of signals, ζ cannot be an estimator of x in the sense of Definitions 10 and 11.

We can now state the optimal two-player controller in the following simple form.

$$u = Kx_{|y_1,u_\zeta} + \begin{bmatrix} 0 & 0 \\ H & J \end{bmatrix} \left(x_{|y,u} - x_{|y_1,u_\zeta}\right) \tag{35}$$

where $H := \hat{K}_{21}$. These equations make clear that Player 1 is using the same control action that it would use if the controller was centralized with both players only measuring y_1. We can also see that the control action of Player 2 has an additional correction term, given by the gain matrix $\begin{bmatrix} H & J \end{bmatrix}$ multiplied by the difference between the players' estimates of the states. Note that u_ζ is a function only of y_1, as can be seen from the state-space equations

$$\begin{aligned} \dot{\zeta} &= A_K\zeta - \hat{L}E_1\left(y_1 - \begin{bmatrix} C_{11} & 0 \end{bmatrix} \zeta\right) \\ u_\zeta &= (K - \hat{K})\zeta \end{aligned}$$

The orthogonality relationships of the form (33) used in Lemma 12 and Theorem 14 to solve the $\mathcal{H}_2$ optimization problems may also be interpreted in terms of signals in the optimally controlled closed-loop.

Corollary 15. *Suppose u, w, x and y satisfy (20)–(24). Then the maps $E_2 : w \to x - \xi$ and $R_2 : w \to y - C_2\xi$, which give the error and residual for Player 2, are*

$$E_2 = \left[\begin{array}{c|c} A_L & B_1 + LD_{21} \\ \hline I & 0 \end{array}\right] \qquad R_2 = \left[\begin{array}{c|c} A_L & B_1 + LD_{21} \\ \hline C_2 & D_{21} \end{array}\right]$$

The maps $E_1 : w \to x - \zeta$ and $R_1 : w \to y_1 - C_{11}\zeta_1$, which give the error and residual for Player 1, are

$$E_1 = \left[\begin{array}{cc|c} \hat{A} & (\hat{L} - L)C_2 & (\hat{L} - L)D_{21} \\ 0 & A_L & B_1 + LD_{21} \\ \hline I & I & 0 \end{array}\right] \qquad R_1 = \left[\begin{array}{c|c} A_M & ME_1^\mathsf{T} - E_1^\mathsf{T}L \\ \hline C_{11} & E_1^\mathsf{T} \end{array}\right] R_2$$

Furthermore, the orthogonality conditions $E_2R_2^* \in \mathcal{H}_2^\perp$ and $E_1R_1^* \in \mathcal{H}_2^\perp$ are satisfied.

As with Theorem 8, Corollary 15 lends itself to a statistical interpretation. If w is white Gaussian noise with unit intensity, and we consider the steady-state distributions of the error and residual for the second player, $x - \xi$ and $y - C_2\xi$ respectively, then they are independent. Similarly, the error and residual for the first player, $x - \zeta$ and $y_1 - C_{11}\zeta_1$, are also independent.

V-D Optimal cost

We now compute the cost associated with the optimal control policy for the two-player output-feedback problem. From centralized $\mathcal{H}_2$ theory [34], there are many equivalent expressions for the optimal centralized cost. In particular,

$$\begin{aligned}\|F_\ell(\mathcal{P},\mathcal{K}_{\text{cen}})\|_2^2 &= \left\| \left[\begin{array}{c|c} A_K & B_1 \\ \hline C_1 + D_{12}K & 0 \end{array}\right] \right\|_2^2 + \left\| \left[\begin{array}{c|c} A_L & B_1 + LD_{21} \\ \hline D_{12}K & 0 \end{array}\right] \right\|_2^2 \\ &= \operatorname{trace}(XW) + \operatorname{trace}(YK^\mathsf{T}RK) \\ &= \operatorname{trace}(YQ) + \operatorname{trace}(XLVL^\mathsf{T})\end{aligned}$$

where X, Y, K, L are defined in (13). Of course, the cost of the optimal two-player controller will be greater, so we have

$$\|F_\ell(\mathcal{P},\mathcal{K}_{\text{opt}})\|_2^2 = \|F_\ell(\mathcal{P},\mathcal{K}_{\text{cen}})\|_2^2 + \Delta$$

where $\Delta \geq 0$ is the additional cost incurred by decentralization. We now give some closed-form expressions for Δ that are similar to the centralized formulae above.

Theorem 16. *The additional cost incurred by the optimal controller* (18) *for the two-player problem* (4) *as compared to the cost of the optimal centralized controller is*

$$\begin{aligned}\Delta &= \left\| \left[\begin{array}{c|c} \hat{A} & (\hat{L} - L)D_{21} \\ \hline D_{12}(\hat{K} - K) & 0 \end{array}\right] \right\|_2^2 \\ &= \operatorname{trace}(\hat{Y} - Y)(\hat{K} - K)^\mathsf{T} R(\hat{K} - K) \\ &= \operatorname{trace}(\hat{X} - X)(\hat{L} - L)V(\hat{L} - L)^\mathsf{T}\end{aligned}$$

where K, $\hat{K}$, L, $\hat{L}$ are defined in (13)–(16)*, $\hat{X}$ and $\hat{Y}$ are defined in* (25) *and* (27)*, and $\hat{A}$ is defined in* (17)*.*

Proof. The key is to view $\mathcal{K}_{\text{opt}}$ as a sub-optimal centralized controller. Centralized $\mathcal{H}_2$ theory [34] then implies that

$$\Delta = \|D_{12}\mathcal{Q}_{\text{you}}D_{21}\|_2^2 \tag{36}$$

where $\mathcal{Q}_{\text{you}}$ is the centralized Youla parameter. Specifically, $\mathcal{Q}_{\text{you}} = F_u(\mathcal{J}^{-1}, \mathcal{K}_{\text{opt}})$ and

$$\mathcal{J}^{-1} = \left[\begin{array}{c|cc} A & B_2 & -L \\ \hline C_2 & 0 & I \\ -K & I & 0 \end{array}\right]$$

This centralized Youla parameterization contains the gains K and L instead of K_d and L_d. After simplifying, we obtain

$$\mathcal{Q}_{\text{you}} = \left[\begin{array}{c|c} \hat{A} & \hat{L} - L \\ \hline \hat{K} - K & 0 \end{array}\right] \tag{37}$$

substituting (37) into (36) yields the first formula. The second formula follows from evaluating (36) in a different way. Note that $\|D_s + C_s(sI - A_s)^{-1}B_s\|^2 = \operatorname{trace}(C_s W_c C_s^{\mathsf{T}})$, where W_c is the associated controllability Gramian, given by $A_s W_c + W_c A_s^{\mathsf{T}} + B_s B_s^{\mathsf{T}} = 0$. In the case of (36), the controllability Gramian equation is precisely (25), and therefore $W_c = \hat{Y} - Y$ and the second formula follows. The third formula follows using the observability Gramian $W_o = \hat{X} - X$ given by (27). ∎

Theorem 16 precisely quantifies the cost of decentralization. We note that Δ is small whenever $(\hat{L} - L)$ or $(\hat{K} - K)$ is small. Examining these cases individually, we find the following. If $L = \hat{L}$, then even the optimal centralized controller makes no use of y_2; it only adds noise and no new information. In the two-player context, this leads to $\zeta = \xi$, so both players have the same estimate of the global state, based only on y_1, and either player is capable of determining u. On the other hand, if $K = \hat{K}$, then even the optimal centralized controller makes no use of u_1. In the two-player context, this leads to a policy in which Player 1 does nothing.

V-E Some special cases

The optimal controller (18) is given by (35), which we repeat here for convenience.

$$u = Kx_{|y_1,u_\zeta} + \begin{bmatrix} 0 & 0 \\ H & J \end{bmatrix} \left(x_{|y,u} - x_{|y_1,u_\zeta}\right)$$

Recall that K and J are found by solving standard AREs (13). The coupling between estimation and control appears in the term $H = -R_{22}^{-1}(B_{22}^{\mathsf{T}}\Phi + S_{12}^{\mathsf{T}})$, which is found by solving the coupled linear equations (14)–(15).

Several special cases of the two-player output-feedback problem have previously been solved. We will see that in each case, the first component of $(x_{|y,u} - x_{|y_1,u_\zeta})$ is zero. In other words, both players maintain identical estimates of x_1 given their respective information. Consequently, u no longer depends on H and there is no need to compute Φ and Ψ. We now examine these special cases in more detail.

Centralized The problem becomes centralized when both players have access to the same information. In this case, both players maintain identical estimates of the entire state x. Thus, $x_{|y,u} = x_{|y_1,u_\zeta} = \hat{x}$ and we recover the well-known centralized solution $u = K\hat{x}$.

State feedback The state-feedback problem for two players is the case where our measurements are noise-free, so that $y_1 = x_1$ and $y_2 = x_2$. Therefore, both players measure x_1 exactly. This case is solved in [24, 25] and the solution takes the following form, which agrees with our general formula.

$$u = K \begin{bmatrix} x_1 \\ \hat{x}_{2|1} \end{bmatrix} + \begin{bmatrix} 0 \\ J \end{bmatrix} (x_2 - \hat{x}_{2|1})$$

Here, $\hat{x}_{2|1}$ is an estimate of x_2 given the information available to Player 1, as stated in [25].

Partial output feedback In the partial output-feedback case, $y_1 = x_1$ as in the state-feedback case, but y_2 is a noisy linear measurement of both states. This case is solved in [26] and the solution takes the following form (using notation from [26]), which agrees with our general formula.

$$u = K \begin{bmatrix} x_1 \\ \hat{x}_{2|1} \end{bmatrix} + \begin{bmatrix} 0 \\ J \end{bmatrix} (\hat{x}_{2|2} - \hat{x}_{2|1})$$

Dynamically decoupled In the dynamically decoupled case, all measurements are noisy, but the dynamics of both systems are decoupled. This amounts to the case where $A_{21} = 0$, $B_{21} = 0$, $C_{21} = 0$, and W, V, U are block-diagonal. Due to the decoupled dynamics, the estimate of x_1 based on y_1 does not improve when additionally using y_2. This case is solved in [6] and the solution takes the following form, which agrees with our general formula.

$$u = K \begin{bmatrix} \hat{x}_{1|1} \\ \hat{x}_{2|1} \end{bmatrix} + \begin{bmatrix} 0 \\ J \end{bmatrix} (\hat{x}_{2|2} - \hat{x}_{2|1})$$

Note that estimation and control are decoupled in all the special cases examined above. This fact allows the optimal controller to be computed by merely solving some subset of the AREs (13). In the general case however, estimation and control are coupled via Φ and Ψ in (14)–(15).

VI Proofs

VI-A Existence and uniqueness of the controller

In order to prove existence and uniqueness, our general approach is to first convert the optimal control problem into a model-matching problem using Corollary 5. This model-matching problem has stable parameters $\mathcal{T}_{ij}$, and so may be solved using standard optimization methods on the Hilbert space $\mathcal{H}_2$. We therefore turn our attention to this class of problems. We will need the following assumptions.

B1) $\mathcal{T}_{11}(\infty) = 0$

B2) $\mathcal{T}_{12}(\mathrm{i}\omega)$ has full column rank for all $\omega \in \mathbb{R} \cup \{\infty\}$

B3) $\mathcal{T}_{21}(\mathrm{i}\omega)$ has full row rank for all $\omega \in \mathbb{R} \cup \{\infty\}$

The optimality condition for centralized model-matching is given in the following lemma.

Lemma 17. *Suppose $\mathcal{T} \in \mathcal{RH}_\infty$ satisfies Assumptions B1–B3. Then the model-matching problem*

$$\begin{aligned} \text{minimize} \quad & \left\| \mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}\mathcal{T}_{21} \right\|_2 \\ \text{subject to} \quad & \mathcal{Q} \in \mathcal{H}_2 \end{aligned} \tag{38}$$

has a unique solution. Furthermore, $\mathcal{Q}$ is the minimizer of (38) *if and only if*

$$\mathcal{T}_{12}^* \left(\mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}\mathcal{T}_{21}\right) \mathcal{T}_{21}^* \in \mathcal{H}_2^\perp \tag{39}$$

Proof. The Hilbert projection theorem (see, for example [14]) states that if H is a Hilbert space, $S \subseteq H$ is a closed subspace, and $b \in H$, then there exists a unique $x \in S$ that minimizes $\|x - b\|_2$. Furthermore, a necessary and sufficient condition for optimality of x is that $x \in S$ and $x - b \in S^\perp$. Given a bounded linear map $A : H \to H$ which is bounded below, define $T := \{Ax \,|\, x \in S\}$. Then T is closed and the projection theorem implies that the problem

$$\begin{aligned} \text{minimize} \quad & \|Ax - b\|_2 \\ \text{subject to} \quad & x \in S \end{aligned}$$

has a unique solution. Furthermore, x is optimal if and only if $x \in S$ and $A^*(Ax-b) \in S^\perp$. This result directly implies the lemma, by setting $H := \mathcal{L}_2$ and $S := \mathcal{H}_2$, and defining the bounded linear operator $A : \mathcal{L}_2 \to \mathcal{L}_2$ by $A\mathcal{Q} := \mathcal{T}_{12}\mathcal{Q}\mathcal{T}_{21}$, with adjoint $A^*\mathcal{P} := \mathcal{T}_{12}^*\mathcal{P}\mathcal{T}_{21}^*$. The operator A is bounded below as a consequence of Assumptions B2–B3. ∎

We now develop an optimality condition similar to (39), but for the two-player structured model-matching problem.

Lemma 18. *Suppose $\mathcal{T} \in \mathcal{RH}_\infty$ satisfies Assumptions B1–B3. Then the two-player model-matching problem*

$$\begin{aligned} &\text{minimize} && \left\|\mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}\mathcal{T}_{21}\right\|_2 \\ &\text{subject to} && \mathcal{Q} \in \operatorname{lower}(\mathcal{H}_2) \end{aligned} \tag{40}$$

has a unique solution. Furthermore, $\mathcal{Q}$ is the minimizer of (40) *if and only if*

$$\mathcal{T}_{12}^*\left(\mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}\mathcal{T}_{21}\right)\mathcal{T}_{21}^* \in \begin{bmatrix} \mathcal{H}_2^\perp & \mathcal{L}_2 \\ \mathcal{H}_2^\perp & \mathcal{H}_2^\perp \end{bmatrix} \tag{41}$$

Proof. The proof follows exactly that of Lemma 17. ∎

Each of these model-matching problems has a rational transfer function solution, as we now show.

Lemma 19. *Suppose $\mathcal{T} \in \mathcal{RH}_\infty$ satisfies Assumptions B1–B3 and has a minimal joint realization given by*

$$\begin{bmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & 0 \end{bmatrix} = \left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & 0 & \bar{D}_{12} \\ \bar{C}_2 & \bar{D}_{21} & 0 \end{array}\right] \quad \text{where } \bar{A} \text{ is Hurwitz.} \tag{42}$$

The solution to the centralized model-matching problem (38) *is rational, and has realization*

$$\mathcal{Q}_{\text{opt}} = \left[\begin{array}{cc|c} \bar{A}_K & \bar{B}_2\bar{K} & 0 \\ 0 & \bar{A}_L & -\bar{L} \\ \hline \bar{K} & \bar{K} & 0 \end{array}\right] \tag{43}$$

where $\bar{K}$, $\bar{L}$ are defined in (13)*, and $\bar{A}_K$, $\bar{A}_L$ are defined in* (17) *with all state-space parameters replaced by their barred counterparts.*

Proof. It is straightforward to verify that Assumptions A1–A6 hold for the realization (42). Optimality is verified by substituting (43) directly into the optimality condition (39) and applying Lemma 17. ∎

Lemma 20. *Suppose $\mathcal{T} \in \mathcal{RH}_\infty$ satisfies Assumptions B1–B3. Then the optimal solution of the two-player model-matching problem* (40) *is rational.*

Proof. Since the $\mathcal{H}_2$-norm is invariant under rearrangement of matrix elements, we may vectorize [5] the contents of the norm in (40) to obtain

$$\begin{aligned} &\text{minimize} && \left\|\operatorname{vec}(\mathcal{T}_{11}) + (\mathcal{T}_{21}^\mathsf{T} \otimes \mathcal{T}_{12}) \operatorname{vec}(\mathcal{Q})\right\|_2 \\ &\text{subject to} && \mathcal{Q} \in \operatorname{lower}(\mathcal{H}_2) \end{aligned} \tag{44}$$

Due to the sparsity pattern of $\mathcal{Q}$, some entries of $\operatorname{vec}(\mathcal{Q})$ will be zero. Let E be the identity matrix with columns removed corresponding to these zero-entries. Then $\mathcal{Q} \in \operatorname{lower}(\mathcal{H}_2)$ if and only if $\operatorname{vec}(\mathcal{Q}) = Eq$ for some $q \in \mathcal{H}_2$. Then (44) is equivalent to

$$\begin{aligned} &\text{minimize} && \left\|\operatorname{vec}(\mathcal{T}_{11}) + (\mathcal{T}_{21}^\mathsf{T} \otimes \mathcal{T}_{12})Eq\right\|_2 \\ &\text{subject to} && q \in \mathcal{H}_2 \end{aligned} \tag{45}$$

This is a centralized model-matching problem of the form (38). Using standard properties of the Kronecker product, one may verify that Assumptions B1–B3 are inherited by

$$\begin{bmatrix} \mathrm{vec}(\mathcal{T}_{11}) & (\mathcal{T}_{21}^{\mathsf{T}} \otimes \mathcal{T}_{12})E \\ 1 & 0 \end{bmatrix} \tag{46}$$

By Lemma 19 the optimal q is rational, and hence the result follows. ■

We now prove that under the assumptions of Theorem 6, the two-player output-feedback problem (4) has a unique solution.

Proof of Theorem 6, Part (i). We apply Corollary 5 to reduce the optimal control problem to the two-player model-matching problem over $\mathcal{RH}_\infty$. It is straightforward to check that under Assumptions A1–A6, the particular $\mathcal{T}$ given in (9) satisfies B1–B3. These assumptions imply that $\mathcal{T}_{12}(\infty)$ is left-invertible and $\mathcal{T}_{21}(\infty)$ is right-invertible. Thus, the optimal solution $\mathcal{Q}_{\text{opt}} \in \mathcal{RH}_\infty$ must have $\mathcal{Q}(\infty) = 0$ to ensure that $\|\mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}\mathcal{T}_{21}\|_2$ is finite. Therefore we may replace the constraint that $\mathcal{Q} \in \mathrm{lower}(\mathcal{RH}_\infty)$ as in Corollary 5 with the constraint that $\mathcal{Q} \in \mathrm{lower}(\mathcal{RH}_2)$ without any loss of generality. Existence and uniqueness now follows from Lemma 18. Rationality follows from Lemma 20. ■

The vectorization approach of Lemma 18 effectively reduces the structured model-matching problem (40) to a centralized model-matching problem, which has a known solution, shown later in Lemma 19. Unfortunately, constructing the solution in this manner is not feasible in practice because it requires finding a state-space realization of the Kronecker system (46). This leads to a dramatic increase in state dimension, and requires solving a large Riccati equation. Furthermore, we lose any physical interpretation of the states, as mentioned in Section I.

VI-B Formulae for the optimal controller

Proof of Theorem 6, Part (ii). Suppose the linear equations (14)–(15) are satisfied by some Φ, Ψ and the proposed $\mathcal{K}_{\text{opt}}$ has been defined according to (18). We will make the following simplifying assumption.

$$L_1 = M \qquad \text{and} \qquad K_2 = J \tag{47}$$

where L_1 and K_2 were originally defined in Lemma 2, and M and J are defined in (13). There is no loss of generality in choosing this particular parameterization for $\mathcal{T}$, but it leads to simpler algebra. To verify optimality of $\mathcal{K}_{\text{opt}}$, we use the parameterization of Theorem 4 to find the $\mathcal{Q}_{\text{opt}}$ that generates $\mathcal{K}_{\text{opt}}$. The computation yields

$$\begin{aligned} \mathcal{Q}_{\text{opt}} &= F_u(\mathcal{J}_d^{-1}, \mathcal{K}_{\text{opt}}) \\ &= \left[\begin{array}{ccc|c} A_K & -\hat{L}C_2 & 0 & \hat{L} \\ 0 & \hat{A} & -B_2\hat{K} & L_d - \hat{L} \\ 0 & 0 & A_L & L_d - L \\ \hline K_d - K & K_d - \hat{K} & \hat{K} & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_K & 0 & \hat{L} \\ 0 & A_L & L_d - L \\ \hline K_d - K & \hat{K} & 0 \end{array}\right] \end{aligned} \tag{48}$$

where A_K, A_L, $\hat{A}$ are defined in (17). The last simplification in (48) comes thanks to (47). The sparsity structure of the gains $\hat{L}$ and $\hat{K}$ gives $\mathcal{Q}_{\text{opt}}$ a block-lower-triangular structure.

Note also that A_K, A_L, $\hat{A}$ are Hurwitz, so $\mathcal{Q}_{\text{opt}} \in \text{lower}(\mathcal{RH}_2)$. It follows from Theorem 4 that $\mathcal{K}_{\text{opt}}$ is an admissible stabilizing controller.

We now directly verify that $\mathcal{Q}_{\text{opt}}$ defined in (48) satisfies the optimality condition (41) for the model-matching problem characterized by the $\mathcal{T}$ given in (9). The closed-loop map $\mathcal{T}_{11}+\mathcal{T}_{12}\mathcal{Q}_{\text{opt}}\mathcal{T}_{21}$ has a particularly nice expression. Substituting in (48) and (9) and simplifying, we obtain

$$\begin{aligned}\mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}_{\text{opt}}\mathcal{T}_{21} &= \left[\begin{array}{c|c} A_{c\ell} & B_{c\ell} \\ \hline C_{c\ell} & 0 \end{array}\right] \\ &= \left[\begin{array}{ccc|c} A_K & -\hat{L}C_2 & 0 & -\hat{L}D_{21} \\ 0 & \hat{A} & -B_2\hat{K} & B_1 + \hat{L}D_{21} \\ 0 & 0 & A_L & B_1 + LD_{21} \\ \hline C_1 + D_{12}K & C_1 + D_{12}\hat{K} & -D_{12}\hat{K} & 0 \end{array}\right]\end{aligned}$$

Note that K_d and L_d are now absent, as the optimal closed-loop map does not depend on the choice of parameterization. The left-hand side of the optimality condition (41) is therefore

$$\mathcal{T}_{12}^*\left(\mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}_{\text{opt}}\mathcal{T}_{21}\right)\mathcal{T}_{21}^* = \left[\begin{array}{ccc|c} -A_{Kd}^{\mathsf{T}} & -C_{Kd}^{\mathsf{T}}C_{c\ell} & 0 & 0 \\ 0 & A_{c\ell} & B_{c\ell}B_{Ld}^{\mathsf{T}} & B_{c\ell}D_{21}^{\mathsf{T}} \\ 0 & 0 & -A_{Ld}^{\mathsf{T}} & -C_2^{\mathsf{T}} \\ \hline B_2^{\mathsf{T}} & D_{12}^{\mathsf{T}}C_{c\ell} & 0 & 0 \end{array}\right]$$

Apply Lemmas 7 and 9 to define $\hat{X}$ and $\hat{Y}$. Now perform the state transformation $x \mapsto Tx$ with

$$T := \begin{bmatrix} I & -\begin{bmatrix} X & \hat{X} & 0 \end{bmatrix} & 0 \\ 0 & I & -\begin{bmatrix} 0 \\ \hat{Y} \\ Y \end{bmatrix} \\ 0 & 0 & I \end{bmatrix}$$

At this point we make use of the Riccati equations (13), as well as the Sylvester equations (14)–(15) via the identities (25)–(28). This leads to a state space realization with $5n$ states, and sparsity pattern of the form

$$\Omega = \left[\begin{array}{ccccc|c} -A_{Kd}^{\mathsf{T}} & 0 & E_1\star & E_1\star & \star & \star \\ 0 & \star & \star & 0 & \star E_2^{\mathsf{T}} & \star \\ 0 & 0 & \hat{A} & \star & \star E_2^{\mathsf{T}} & \star E_2^{\mathsf{T}} \\ 0 & 0 & 0 & \star & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_{Ld}^{\mathsf{T}} & -C_2^{\mathsf{T}} \\ \hline B_2^{\mathsf{T}} & 0 & E_1\star & \star & \star & 0 \end{array}\right]$$

where $\star$ denotes a matrix whose value is unimportant. The second and fourth states are unobservable and uncontrollable, respectively. Removing them, we are left with

$$\Omega = \left[\begin{array}{ccc|c} -A_{Kd}^{\mathsf{T}} & E_1\star & \star & \star \\ 0 & \hat{A} & \star E_2^{\mathsf{T}} & \star E_2^{\mathsf{T}} \\ 0 & 0 & -A_{Ld}^{\mathsf{T}} & -C_2^{\mathsf{T}} \\ \hline B_2^{\mathsf{T}} & E_1\star & \star & 0 \end{array}\right]$$

Because of the block-triangular structure of A, B_2, C_2 and the block-diagonal structure of K_d, L_d, it is straightforward to check that

$$\Omega E_1 = \left[\begin{array}{cc|c} -A_{Kd}^{\mathsf{T}} & \star & \star \\ 0 & -A_M^{\mathsf{T}} & -C_{11}^{\mathsf{T}} \\ \hline B_2^{\mathsf{T}} & \star & 0 \end{array}\right] \in \mathcal{H}_2^{\perp}$$

Similarly, $E_2^\mathsf{T}\Omega \in \mathcal{H}_2^\perp$, and therefore

$$\mathcal{T}_{12}^* \left(\mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}_{\text{opt}}\mathcal{T}_{21}\right)\mathcal{T}_{21}^* \in \begin{bmatrix} \mathcal{H}_2^\perp & \mathcal{L}_2 \\ \mathcal{H}_2^\perp & \mathcal{H}_2^\perp \end{bmatrix}$$

So by Lemma 18, $\mathcal{Q}_{\text{opt}}$ is the solution to the model-matching problem (40), and therefore must also be a minimizer of (11). It follows from Corollary 5 that $\mathcal{K}_{\text{opt}}$ is the unique optimal controller for the two-player output-feedback problem (4). ∎

VI-C Existence of solutions to the Sylvester equations

We now show that the linear equations (14)–(15) have a solution. First, we show that existence of the optimal controller implies a certain fixed point property. A simple necessary condition for optimality is person-by-person optimality; by fixing any part of the optimal $\mathcal{Q}$ and optimizing over the rest, we cannot improve upon the optimal cost.

Lemma 21. *Suppose $\mathcal{T}$ is given by* (9) *and Assumptions A1–A6 hold. Further suppose that $\mathcal{Q} \in \operatorname{lower}(\mathcal{RH}_2)$ and*

$$\mathcal{Q} := \begin{bmatrix} \mathcal{Q}_{11} & 0 \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix}$$

Define the following partial optimization functions.

$$g_1(\mathcal{Q}_{11}) := \underset{\mathcal{Q}_{22}\in\mathcal{RH}_2}{\arg\min} \left(\min_{\mathcal{Q}_{21}\in\mathcal{RH}_2} \left\| \mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}\mathcal{T}_{21} \right\|_2 \right) \tag{49}$$

$$g_2(\mathcal{Q}_{22}) := \underset{\mathcal{Q}_{11}\in\mathcal{RH}_2}{\arg\min} \left(\min_{\mathcal{Q}_{21}\in\mathcal{RH}_2} \left\| \mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}\mathcal{T}_{21} \right\|_2 \right) \tag{50}$$

If $\mathcal{Q}$ is optimal for the two-player model-matching problem (40)*, then*

$$g_2(g_1(\mathcal{Q}_{11})) = \mathcal{Q}_{11} \qquad \textit{and} \qquad g_1(g_2(\mathcal{Q}_{22})) = \mathcal{Q}_{22}$$

Proof. Under the stated assumptions, the optimization problems (49)–(50) have unique optimal solutions. If $\mathcal{Q}$ is optimal, then clearly we have $\mathcal{Q}_{22} = g_1(\mathcal{Q}_{11})$ and $\mathcal{Q}_{11} = g_2(\mathcal{Q}_{22})$. The result follows by substituting one identity into the other. ∎

An alternative way of stating Lemma 21 is that the optimal $\mathcal{Q}_{11}$ and $\mathcal{Q}_{22}$ are the fixed points of the maps $g_2 \circ g_1$ and $g_1 \circ g_2$ respectively. Our next step is to solve these fixed-point equations analytically. The key insight is that (49)–(50) are centralized model-matching problems of the form (38). In the following lemma, we fix $\mathcal{Q}_{11}$ and we find $\begin{bmatrix} \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix}$ that minimizes the right-hand side of (49).

Lemma 22. *Assume $\mathcal{T} \in \mathcal{RH}_\infty$ is given by* (9)*. Suppose Assumptions A1–A6 hold together with the structural requirement* (6) *and the parameter choice* (47)*. If*

$$\mathcal{Q}_{11} := \left[\begin{array}{c|c} A_P & B_P \\ \hline C_P & 0 \end{array}\right] \qquad \textit{where } A_P \textit{ is Hurwitz}$$

then the right-hand side of (49) *is minimized by*

$$\begin{bmatrix} \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} = \left[\begin{array}{ccc|c} A_{Kd} + B_2 \begin{bmatrix} 0 \\ \bar{K}_1 \end{bmatrix} & B_2 \begin{bmatrix} 0 \\ \bar{K}_2 \end{bmatrix} & B_2 \begin{bmatrix} C_P \\ \bar{K}_3 \end{bmatrix} & -L_d \\ 0 & A_L & 0 & L_d - L \\ 0 & 0 & A_P & B_P E_1^\mathsf{T} \\ \hline \bar{K}_1 & \bar{K}_2 & \bar{K}_3 & 0 \end{array}\right] \tag{51}$$

Here the quantities $\bar{K}_1$, $\bar{K}_2$, and $\bar{K}_3$ are defined by

$$\begin{gathered}\bar{K}_1 = \begin{bmatrix}-R_{22}^{-1}\left(B_{22}^\mathsf{T}\Theta_X + S_{12}^\mathsf{T} + R_{12}^\mathsf{T}K_1\right) & 0\end{bmatrix} \qquad \bar{K}_2 = \begin{bmatrix}-R_{22}^{-1}\left(B_{22}^\mathsf{T}\Phi + S_{12}^\mathsf{T}\right) & J\end{bmatrix} \\ \bar{K}_3 = -R_{22}^{-1}\left(B_{22}^\mathsf{T}\Gamma_X + R_{12}^\mathsf{T}C_P\right)\end{gathered} \tag{52}$$

where Θ_X, Γ_X, and Φ are the unique solutions to the linear equations

$$\begin{aligned} A_J^\mathsf{T}\Theta_X + \begin{bmatrix}\Theta_X & \tilde{X}\end{bmatrix} A_{Kd}E_1 + E_2^\mathsf{T}C_{Kd}^\mathsf{T}C_{Kd}E_1 &= 0 \\ A_J^\mathsf{T}\Gamma_X + \Gamma_X A_P + \left(\begin{bmatrix}\Theta_X & \tilde{X}\end{bmatrix} B_2E_1 + J^\mathsf{T}R_{21} + S_{21}\right)C_P &= 0 \\ A_J^\mathsf{T}\Phi + \Phi A_M + \tilde{X}A_{21} - \Theta_X M C_{11} + J^\mathsf{T}S_{12}^\mathsf{T} + Q_{21} + \Gamma_X B_P C_{11} &= 0 \end{aligned} \tag{53}$$

where Q, R, S are defined in (12)*, $\tilde{X}, Y, J, L$ are defined in* (13)*, and A_{Kd}, C_{Kd} are defined in* (10).

Proof. Since $\mathcal{Q}_{11}$ is held fixed, group it with $\mathcal{T}_{11}$ to obtain

$$\mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}\mathcal{T}_{21} = \left(\mathcal{T}_{11} + \mathcal{T}_{12}E_1\mathcal{Q}_{11}E_1^\mathsf{T}\mathcal{T}_{21}\right) + \mathcal{T}_{12}E_2\begin{bmatrix}\mathcal{Q}_{21} & \mathcal{Q}_{22}\end{bmatrix}\mathcal{T}_{21}$$

This is an affine function of $\begin{bmatrix}\mathcal{Q}_{21} & \mathcal{Q}_{22}\end{bmatrix}$, so the associated model-matching problem is centralized. Finding a joint realization of the blocks as in Lemma 19, we obtain

$$\begin{aligned}\begin{bmatrix}\mathcal{T}_{11} + \mathcal{T}_{12}E_1\mathcal{Q}_{11}E_1^\mathsf{T}\mathcal{T}_{21} & \mathcal{T}_{12}E_2 \\ \mathcal{T}_{21} & 0\end{bmatrix} &= \left[\begin{array}{c|cc}\bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & 0 & \bar{D}_{12} \\ \bar{C}_2 & \bar{D}_{21} & 0\end{array}\right] \\ &= \left[\begin{array}{ccc|cc} A_{Kd} & -L_dC_2 & B_2E_1C_P & -L_dD_{21} & B_2E_2 \\ 0 & A_{Ld} & 0 & B_{Ld} & 0 \\ 0 & B_PE_1^\mathsf{T}C_2 & A_P & B_PE_1^\mathsf{T}D_{21} & 0 \\ \hline C_{Kd} & C_1 & D_{12}E_1C_P & 0 & D_{12}E_2 \\ 0 & C_2 & 0 & D_{21} & 0 \end{array}\right]\end{aligned}$$

It is straightforward to check that Assumptions B1–B3 are satisfied for this augmented system. Now, we may apply Lemma 19. The result is that

$$\begin{bmatrix}\mathcal{Q}_{21} & \mathcal{Q}_{22}\end{bmatrix}_{\text{opt}} = \left[\begin{array}{cc|c}\bar{A} + \bar{B}_2\bar{K} & \bar{B}_2\bar{K} & 0 \\ 0 & \bar{A} + \bar{L}\bar{C}_2 & -\bar{L} \\ \hline \bar{K} & \bar{K} & 0\end{array}\right] \tag{54}$$

where we defined $(\bar{X}, \bar{K}) := \operatorname{ARE}(\bar{A}, \bar{B}_2, \bar{C}_1, \bar{D}_{12})$ and $(\bar{Y}, \bar{L}^\mathsf{T}) := \operatorname{ARE}(\bar{A}^\mathsf{T}, \bar{C}_2^\mathsf{T}, \bar{B}_1^\mathsf{T}, \bar{D}_{21}^\mathsf{T})$. One can check that the stabilizing solution to the latter ARE is

$$\bar{Y} = \begin{bmatrix}0 & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & 0\end{bmatrix} \qquad \text{and} \qquad \bar{L} = \begin{bmatrix}L_d \\ L - L_d \\ -B_PE_1^\mathsf{T}\end{bmatrix}$$

The former ARE is more complicated, however. Examining $\bar{K} = -\bar{R}^{-1}(\bar{B}^\mathsf{T}\bar{X} + \bar{S}^\mathsf{T})$, we notice that due to all the zeros in $\bar{B}$, the only part of $\bar{X}$ that affects the gain $\bar{K}$ is the second sub-row of the first block-row. In other words, if

$$\bar{X} := \begin{bmatrix}\bar{X}_{11} & \bar{X}_{12} & \bar{X}_{13} \\ \bar{X}_{21} & \bar{X}_{22} & \bar{X}_{23} \\ \bar{X}_{31} & \bar{X}_{32} & \bar{X}_{33}\end{bmatrix}$$

then $\bar{K}$ only depends on $E_2^{\mathsf{T}}\bar{X}_{11}$, $E_2^{\mathsf{T}}\bar{X}_{12}$, and $E_2^{\mathsf{T}}\bar{X}_{13}$. Multiplying the ARE for $\bar{X}$ on the left by $\begin{bmatrix} E_2^{\mathsf{T}} & 0 & 0 \end{bmatrix}$ and on the right by $\begin{bmatrix} E_2^{\mathsf{T}} & 0 & 0 \end{bmatrix}^{\mathsf{T}}$, we obtain the equation

$$A_J^{\mathsf{T}}\tilde{X} + \tilde{X}A_J + (C_1E_2 + D_{12}E_2J)^{\mathsf{T}}(C_1E_2 + D_{12}E_2J) = 0$$

where $\tilde{X} := E_2^{\mathsf{T}}\bar{X}_{11}E_2$. It is straightforward to see that $\tilde{X}$ as defined in (13) satisfies this equation. Substituting this back into the ARE for $\bar{X}$ and multiplying on the left by $\begin{bmatrix} E_2^{\mathsf{T}} & 0 & 0 \end{bmatrix}$, we obtain

$$\begin{aligned} A_J^{\mathsf{T}} &\begin{bmatrix} E_2^{\mathsf{T}}\bar{X}_{11} & E_2^{\mathsf{T}}\bar{X}_{12} & E_2^{\mathsf{T}}\bar{X}_{13} \end{bmatrix} \\ &+ \begin{bmatrix} E_2^{\mathsf{T}}\bar{X}_{11} & E_2^{\mathsf{T}}\bar{X}_{12} & E_2^{\mathsf{T}}\bar{X}_{13} \end{bmatrix} \begin{bmatrix} A_{Kd} & -L_dC_2 & B_2E_1C_P \\ 0 & A_{Ld} & 0 \\ 0 & B_PE_1^{\mathsf{T}}C_2 & A_P \end{bmatrix} \\ &+ (C_1E_2 + D_{12}E_2J)^{\mathsf{T}} \begin{bmatrix} C_{Kd} & C_1 & D_{12}E_1C_P \end{bmatrix} = 0 \end{aligned} \tag{55}$$

Right-multiplying (55) by $\begin{bmatrix} 0 & E_2^{\mathsf{T}} & 0 \end{bmatrix}^{\mathsf{T}}$, we conclude that $E_2^{\mathsf{T}}\bar{X}_{12}E_2 = \tilde{X}$. Notice that (55) is linear in the $\bar{X}$ terms. Assign the following names to the missing pieces

$$E_2^{\mathsf{T}}\bar{X}_{11} := \begin{bmatrix} \Theta_X & \tilde{X} \end{bmatrix} \qquad E_2^{\mathsf{T}}\bar{X}_{12} := \begin{bmatrix} \Phi & \tilde{X} \end{bmatrix} \qquad E_2^{\mathsf{T}}\bar{X}_{13} := \Gamma_X$$

Upon substituting these definitions into (55) and simplifying, we obtain (53). A similar substitution into the definition of $\bar{K} = \begin{bmatrix} \bar{K}_1 & \bar{K}_2 & \bar{K}_3 \end{bmatrix}$ leads to the formulae (52).

The equations (53) have a unique solution. To see why, note that they may be sequentially solved: for Ω_X, then for Γ_X, and finally for Φ. Each is a Sylvester equation of the form $A_1\Omega + \Omega A_2 + A_0 = 0$, where A_1 and A_2 are Hurwitz, so the solution is unique. Furthermore, $\bar{A} + \bar{B}\bar{K}$ is easily verified to be Hurwitz, so the procedure outlined above produces the correct stabilizing $\bar{X}$. Now, substitute $\bar{K}$ and $\bar{L}$ into (54). The result is a very large state-space realization, but it can be greatly reduced by eliminating uncontrollable and unobservable states. The result is (51). This reduction is not surprising, because we solved a model-matching problem in which the joint realization for the three blocks had a zero as the fourth block. ∎

We may solve (50) in a manner analogous to how we solved (49). Namely, we can provide a formula for the optimal $\mathcal{Q}_{11}$ and $\mathcal{Q}_{21}$ as functions of $\mathcal{Q}_{22}$. The result follows directly from Lemma 22 after we make a change of variables.

Lemma 23. *Assume $\mathcal{T} \in \mathcal{RH}_\infty$ is given by* (9)*. Suppose Assumptions A1–A6 hold together with the structural requirement* (6) *and the parameter choice* (47)*. If*

$$\mathcal{Q}_{22} := \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & 0 \end{array}\right] \qquad \text{where } A_Q \text{ is Hurwitz}$$

then the right-hand side of (50) *is minimized by*

$$\begin{bmatrix} \mathcal{Q}_{11} \\ \mathcal{Q}_{21} \end{bmatrix} = \left[\begin{array}{ccc|c} A_{Ld} + \begin{bmatrix} \bar{L}_1 & 0 \end{bmatrix} C_2 & 0 & 0 & \bar{L}_1 \\ \begin{bmatrix} \bar{L}_2 & 0 \end{bmatrix} C_2 & A_K & 0 & \bar{L}_2 \\ \begin{bmatrix} \bar{L}_3 & B_Q \end{bmatrix} C_2 & 0 & A_Q & \bar{L}_3 \\ \hline -K_d & K_d - K & E_2C_Q & 0 \end{array}\right] \tag{56}$$

The quantities $\bar{L}_1$, $\bar{L}_2$, and $\bar{L}_3$ are defined by

$$\bar{L}_1 = \begin{bmatrix} 0 \\ -(\Theta_Y C_{11}^{\mathsf{T}} + U_{12}^{\mathsf{T}} + L_2V_{12}^{\mathsf{T}})V_{11}^{-1} \end{bmatrix} \qquad \bar{L}_2 = \begin{bmatrix} M \\ -(\Psi C_{11}^{\mathsf{T}} + U_{12}^{\mathsf{T}})V_{11}^{-1} \end{bmatrix}$$
$$\bar{L}_3 = -\left(\Gamma_Y C_{11}^{\mathsf{T}} + B_QV_{21}\right)V_{11}^{-1}$$

where Θ_Y, Γ_Y, and Ψ are the unique solutions to the linear equations

$$\begin{aligned} E_2^\mathsf{T} A_{Ld} \begin{bmatrix} \tilde{Y} \\ \Theta_Y \end{bmatrix} + \Theta_Y A_M^\mathsf{T} + E_2^\mathsf{T} B_{Ld} B_{Ld}^\mathsf{T} E_1 &= 0 \\ A_Q \Gamma_Y + \Gamma_Y A_M^\mathsf{T} + B_Q \left(E_2^\mathsf{T} C_2 \begin{bmatrix} \tilde{Y} \\ \Theta_Y \end{bmatrix} + V_{21} M^\mathsf{T} + U_{21} \right) &= 0 \\ A_J \Psi + \Psi A_M^\mathsf{T} + A_{21} \tilde{Y} - B_{22} J \Theta_Y + U_{12}^\mathsf{T} M^\mathsf{T} + W_{21} + B_{22} C_Q \Gamma_Y &= 0 \end{aligned} \tag{57}$$

where W, V, U are defined in (12)*, $X, \tilde{Y}, K, M$ are defined in* (13)*, and A_{Ld}, B_{Ld} are defined in* (10)*.*

A key observation that greatly simplifies the extent to which the optimization problems (49) and (50) are coupled is that the $\mathcal{Q}_{ii}$ have simple state-space representations. In fact, we have the strong conclusion that for any fixed $\mathcal{Q}_{11}$, the optimal $\mathcal{Q}_{22} = g_1(\mathcal{Q}_{11})$ has a fixed state dimension no greater than the dimension of the plant, as in the following result.

Theorem 24. *Suppose $\mathcal{T} \in \mathcal{RH}_\infty$ is given by* (9) *and Assumptions A1–A6 hold together with the structural requirement* (6) *and the parameter choice* (47)*. The functions g_i defined in* (49)*–*(50) *are given by*

$$g_1(\mathcal{Q}_{11}) = \left[\begin{array}{c|c} A_L & (L_d - L)E_2 \\ \hline \bar{K}_2 & 0 \end{array}\right] \qquad g_2(\mathcal{Q}_{22}) = \left[\begin{array}{c|c} A_K & \bar{L}_2 \\ \hline E_1^\mathsf{T}(K_d - K) & 0 \end{array}\right] \tag{58}$$

where $\bar{K}_2$, $\bar{L}_2$ are defined in Lemmas 22 and 23 respectively.

Proof. This follows directly from Lemmas 22 and 23 and some simple state-space manipulations. Note that $g_2(\mathcal{Q}_{22})$ depends on $\mathcal{Q}_{22}$ only through the realization-independent quantities $C_Q A_Q^k B_Q$ for $k \geq 0$, and similarly for g_1. ∎

Proof of Theorem 6, Part (iii). Applying the fixed-point results of Lemma 21, there must exist matrices A_P, B_P, C_P, A_Q, B_Q, C_Q such that

$$\left[\begin{array}{c|c} A_P & B_P \\ \hline C_P & 0 \end{array}\right] = \left[\begin{array}{c|c} A_K & \bar{L}_2 \\ \hline E_1^\mathsf{T}(K_d - K) & 0 \end{array}\right] \tag{59}$$

$$\left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & 0 \end{array}\right] = \left[\begin{array}{c|c} A_L & (L_d - L)E_2 \\ \hline \bar{K}_2 & 0 \end{array}\right] \tag{60}$$

where (53) and (57) are satisfied. Given A_Q, B_Q, C_Q we define Ψ according to (57), and given A_P, B_P, C_P we define Φ according to (53). We will show that these Φ and Ψ satisfy the Sylvester equations (14)–(15).

For convenience, define $\mathcal{S} := (A_P, B_P, C_P, A_Q, B_Q, C_Q)$. Our goal is to use (59)–(60) to eliminate the matrices in $\mathcal{S}$ from (53) and (57). We begin with (59).

Note that we cannot simply set the corresponding state-space parameters in (59) equal to one another. This approach is erroneous because transfer function equality does not in general imply that the state-space matrices are also equal. For example, if $E_1^\mathsf{T}(K_d - K) = 0$, then we can set $C_P = 0$ and any choice of B_P satisfies (59). Equality of transfer functions does however imply equality of the Markov parameters. Namely,

$$C_P A_P^k B_P = E_1^\mathsf{T}(K_d - K) A_K^k \bar{L}_2 \quad \text{for } k = 0, 1, \ldots \tag{61}$$

Now consider (53). Note that Θ_X does not depend on $\mathcal{S}$. Furthermore, the equation for Γ_X is a Sylvester equation of the form $A_J^\mathsf{T}\Gamma_X + \Gamma_X A_P + \Omega C_P = 0$, where

$$\Omega := \begin{bmatrix}\Theta_X & \tilde{X}\end{bmatrix} B_2 E_1 + J^\mathsf{T} R_{21} + S_{21}$$

and Ω is independent of $\mathcal{S}$. Since A_J and A_P are Hurwitz by construction, the unique Γ_X is given by the integral

$$\Gamma_X = \int_0^\infty \exp(A_J^\mathsf{T} t)\, \Omega\, C_P \exp(A_P t)\, \mathrm{d}t$$

Substitute the Markov parameters (61), and conclude that

$$\Gamma_X B_P = \int_0^\infty \exp(A_J^\mathsf{T} t)\, \Omega\, E_1^\mathsf{T}(K_d - K)\exp(A_K t)\bar{L}_2\, \mathrm{d}t$$

The equation above is of the form $\Gamma_X B_P = \hat{\Gamma}_X \bar{L}_2$, where $\hat{\Gamma}_X$ satisfies

$$A_J^\mathsf{T}\hat{\Gamma}_X + \hat{\Gamma}_X A_K + \Omega E_1^\mathsf{T}(K_d - K) = 0 \tag{62}$$

One can verify by direct substitution that (62) is solved by

$$\hat{\Gamma}_X = \begin{bmatrix}\Theta_X - X_{21} & \tilde{X} - X_{22}\end{bmatrix} \tag{63}$$

Noting that the term $\Gamma_X B_P$ appears explicitly in the Φ-equation of (53), we may replace the $\Gamma_X B_P$ term by $\hat{\Gamma}_X \bar{L}_2$, and substitute the expressions for $\hat{\Gamma}_X$ and $\bar{L}_2$ from (63) and Lemma 22, respectively. Doing so, we find that Θ_X cancels. The result is an equation involving only Φ and Ψ, and it turns out to be (14).

Repeating a similar procedure as above by instead examining (60) and (57), we find that $C_Q \Gamma_Y = \bar{K}_2 \hat{\Gamma}_Y$ where $\hat{\Gamma}_Y$ is given by

$$\hat{\Gamma}_Y = \begin{bmatrix}\tilde{Y} - Y_{11} \\ \Theta_Y - Y_{21}\end{bmatrix}$$

The result is a different equation involving only Φ and Ψ. This time, it turns out to be (15).

We have therefore shown that existence of a solution to the fixed-point equations of Lemma 21 implies the existence of a solution to the Sylvester equations (14)–(15). This completes the proof. ■

VII Summary

In this article, we studied the class of two-player output-feedback problems with a nested information pattern.

We began by giving necessary and sufficient state-space conditions for the existence of a structured stabilizing controller. This led to a Youla-like parameterization of all such controllers and a convexification of the two-player problem.

The main result of this paper is explicit state-space formulae for the optimal $\mathcal{H}_2$ controller for the two-player output-feedback problem. In the centralized case, it is a celebrated and widely-generalized result that the controller is a composition of an optimal state-feedback gain with a Kalman filter estimator. Our approach generalizes both the centralized formulae and this separation structure to the two-player decentralized case.

We show that the $\mathcal{H}_2$-optimal structured controller has generically twice the state dimension of the plant, and we give intuitive interpretations for the states of the controller as steady-state Kalman filters. The player with more information must duplicate the estimator of the player with less information. This has the simple anthropomorphic interpretation that Player 2 is correcting mistakes made by Player 1.

Both the state-space dimension and separation structure of the optimal controller were previously unknown. While these results show that the optimal controller for this problem has an extremely simple state-space structure, not all such decentralized problems exhibit such pleasant behavior. One example is the two-player fully-decentralized state-feedback problem, where even though the optimal linear controller is known to be rational, it is shown in [10] that the number of states of the controller may grow quadratically with the state-dimension of the plant.

The formulae that we give for the optimal controller are simply computable, requiring the solution of four standard AREs, two that have the same dimension as the plant and two with a smaller dimension. In addition, one must solve a linear matrix equation. All of these computations are simple and have readily available existing code.

While there is as yet no complete state-space theory for decentralized control, in this work we provide solutions to a prototypical class of problems which exemplify many of the features found in more general problems. It remains a fundamental and important problem to fully understand the separation structure of optimal decentralized controllers in the general case. While we solve the two-player triangular case, we hope that the solution gives some insight and possible hints regarding the currently unknown structure of the optimal controllers for more general architectures.

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