Generalized Necessary and Sufficient Robust Boundedness Results for Feedback Systems

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Abstract

Classical conditions for ensuring the robust stability of a system in feedback with a nonlinearity include passivity, small gain, circle, and conicity theorems. We present a generalized and unified version of these results in an arbitrary semi-inner product space, which avoids many of the technicalities that arise when working in traditional extended spaces. Our general formulation clarifies when the sufficient conditions for robust stability are also necessary, and we show how to construct worst-case scenarios when the sufficient conditions fail to hold. Finally, we show how our general result can be specialized to recover a wide variety of existing results, and explain how properties such as boundedness, causality, linearity, and time-invariance emerge as a natural consequence.

1 Introduction

Robust stability of interconnected systems has been a topic of interest for over 75 years, dating back to the seminal works of Lur'e [16], Zames [34, 35], and Willems [32]. The standard input-output setup is illustrated in Fig. 1, where systems G and Φ are connected in feedback, and we seek conditions under which we can ensure the stability of the closed-loop map $(u_1, u_2) \to (y_1, y_2)$.

Robust stability is usually stated as a sufficient condition. For example: "Suppose $G \in S_1$ and $\Phi \in S_2$. If a certain condition holds on S_1 and S_2, then the interconnection of Fig. 1 is stable." In typical usage, a known system G is interconnected with some unknown, uncertain, or otherwise troublesome nonlinearity $\Phi \in S_2$, where S_2 is given. Then, ensuring stability of the interconnected system for any $\Phi \in S_2$ amounts to verifying the condition $G \in S_1$.

There are many robust stability results in the literature: passivity theory, the small-gain theorem, the circle criterion, graph separation, conic sector theorems, multiplier theory, dissipativity theory, and integral quadratic constraints.[1]

The aforementioned results provide *sufficient* conditions for robust stability. To reduce conservatism, one may ask conditions for robust stability that are *necessary and sufficient.* Such results typically take the form: "Suppose S_1 and S_2 satisfy a certain condition. Then, $G \in S_1$ if and only if the interconnection of Fig. 1 is stable for all $\Phi \in S_2$." Again, there are numerous examples in the literature, such as necessary and sufficient versions of the passivity, small-gain, and circle theorems.[1]

The reason for the wide variety of robust stability results is that different assumptions can be made about G and Φ, and the sets S_1 and S_2 can be defined in many different ways. A natural

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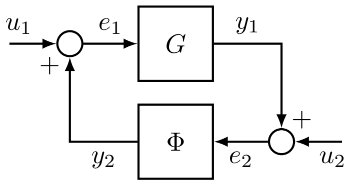
[1] Detailed references can be found in Section 1.1 and Table 1.



$$e_1 = u_1 + y_2 \tag{1a}$$
$$y_2 = \Phi e_2 \tag{1b}$$
$$e_2 = u_2 + y_1 \tag{1c}$$
$$y_1 = G e_1 \tag{1d}$$

Figure 1: Feedback interconnection of systems G and Φ.

question to ask, which forms the basis of our present work, is whether the multitude of existing results can be viewed as consequences of a single more general result. We answer in the affirmative.

Main contributions: In Section 2, we present a robust boundedness result involving interconnected *relations* over a general *semi-inner product space*, where there need not exist a notion of time. Our result (Theorem 1) does not assume linearity or even boundedness of G or Φ, and avoids the technicalities typically associated with *causality*, *stability*, and *well-posedness*. Under mild conditions, our sufficient condition for robust boundedness is also necessary, and we provide a constructive proof.

In Section 3, we specialize our result to standard extended spaces of time-domain signals (e.g., $L_{2\mathrm{e}}$ or $\ell_{2\mathrm{e}}$), which reveals the way in which the aforementioned technicalities arise. We also explain why necessity is more difficult to achieve in this setting, and why stronger assumptions (e.g., linearity and time-invariance of G) are often required.

A key observation, explained in Section 4, is that *sufficient-only* results are often[2] a direct consequence of a corresponding necessary-and-sufficient counterpart, meaning that there is nothing to be gained by stating a sufficient-only version.

1.1 Related work

In Table 1, we provide a summary of existing robust stability results. In the "Implication" column, we distinguish between *sufficient-only* results ($\Longrightarrow$) and *necessary-and-sufficient* results ($\Longleftrightarrow$).

Sufficient results Classical sufficient results include the passivity, small-gain, and circle theorems[3]. These results are mutually related via a loop-shifting transformation [1], and were generalized to *conic sectors* [2, 34, 35].

Beyond conic sector constraints, graph separation [24, 28] allows for nonlinear constraints, while multiplier theory [9], dissipativity [32], and integral quadratic constraints (IQCs) [20, 22, 30] allow for dynamic or time-varying constraints. There have also been several works discussing how these various frameworks are related [5, 10, 25].

Necessary and sufficient results The classical passivity, small-gain, and circle theorems are only sufficient when Φ is assumed to be memoryless (but still possibly time-varying) [4, 18]. However, necessity holds if Φ is allowed to have memory, e.g. when Φ is a dynamical system [31, Thm. 6.6.126].

The majority of necessary and sufficient robust stability results assume that G is linear. For example, the passivity and small gain results of Vidyasagar [31, §6.6(112,126)] and Khong et al. [15, Thm. 3] assume G is linear and time-invariant (LTI). Meanwhile, the small gain result of Zhou et al. [36, Thm. 9.1] and the recent converse IQC result of Khong et al. [14] make the stronger

[2] We make this claim for results involving conicity constraints. These results include: passivity, small-gain, circle criterion, conicity, and extended conicity. See Section 1.1 and Table 1 for details.

[3] The circle criterion in the literature typically refers to the case when G in Figure 1 is assumed to be an LTI system.

Table 1: Literature Review of robust stability results involving an interconnection of two systems (refer to Fig. 1). The first group of rows are sufficient-only results (Implication: $\Longrightarrow$). The second group of rows are necessary and sufficient (Implication: $\Longleftrightarrow$). For constraints on $G \in \mathcal{C}_G$ and $\Phi \in \mathcal{C}_\Phi$ (refer to Fig. 1) we denote linear (L), nonlinear (NL), time-varying (TV), time-invariant (TI), static (S), and fading-memory (FM). For example, "NLTV" indicates nonlinear and time-varying. For vector spaces, the symbols $L_{2\mathrm{e}}$, $\ell_{2\mathrm{e}}$, $\mathcal{L}_{2\mathrm{e}}$ denote extended spaces (see Section 3.1) and s.i.p.s. denotes a semi-inner product space. The final column indicates whether the proof of the converse direction ($\Longleftarrow$), if applicable, explicitly constructs a worst-case Φ when the conditions on G are violated.

Reference	**Result Type**	$\mathcal{C}_G$	$\mathcal{C}_\Phi$	**Implication**	**Vector Space**	**Constructive?**
Vidyasagar [31, §6.6.(1,58)]	passivity & small gain	NLTV	NLTV	$\Longrightarrow$	$\mathcal{L}_{2\mathrm{e}}$	
Zames [34, Thm. 1–3]	conic	NLTV	NLTV	$\Longrightarrow$	$\mathcal{L}_{2\mathrm{e}}$	
Bridgeman & Forbes [3]	conic	NLTV	NLTV	$\Longrightarrow$	$\mathcal{L}_{2\mathrm{e}}$	
Zames [35, §3–4]	circle & multipliers	LTI	NLTVS	$\Longrightarrow$	$L_{2\mathrm{e}}$	
Desoer & Vidyasagar [9]	multipliers	NLTV	NLTV	$\Longrightarrow$	$\mathcal{L}_{2\mathrm{e}}$	
Teel et al. [28]	graph separation	NLTV	NLTV	$\Longrightarrow$	$\mathcal{L}_{2\mathrm{e}}$	
Willems [32]	dissipativity	NLTV	NLTV	$\Longrightarrow$	$\mathcal{L}_{2\mathrm{e}}$	
Pfifer & Seiler [22]	dissipativity	LTI	NLTV	$\Longrightarrow$	$L_{2\mathrm{e}}$	
Megretski & Rantzer [20]	IQC	LTI	NLTVS	$\Longrightarrow^\dagger$	$L_{2\mathrm{e}}$	
Vidyasagar [31, §6.6.(112,126)]	small gain & circle	LTI	NLTV	$\Longleftrightarrow$	$L_{2\mathrm{e}}$	Yes
Khong & van der Schaft [15, Thm. 3]	passivity & small gain	LTI	LTV	$\Longleftrightarrow$	$L_{2\mathrm{e}}$	No
Zhou et al. [36, Thm. 9.1]	small gain	LTI	LTI	$\Longleftrightarrow$	$L_{2\mathrm{e}}$	Yes
Khong & Kao [14, Thm. 1]	IQC	LTI	LTI	$\Longleftrightarrow$	$L_{2\mathrm{e}}$	Yes
Shamma [26, Thm. 3.2]	small gain	NLTVFM	NLTVFM	$\Longleftrightarrow$	$\ell_{2\mathrm{e}}$	Yes
Cyrus & Lessard [8]	conic	L	NL	$\Longleftrightarrow$	s.i.p.s.	No
Present work	conic	NL	NL	$\Longleftrightarrow$	s.i.p.s.	Yes

† The authors in [20] mention that their sufficient condition for robust stability is also necessary in the sense that a result in spirit of Lemma 7 holds via a suitable application of the S-lemma [21].

assumption that *both* G and Φ are LTI. Finally, Shamma's small gain theorem [26, Thm. 3.2] holds when both G and Φ are nonlinear and time-varying, but requires an additional assumption of *fading memory*, which effectively allows the system response to be approximated by that of a linear system.

One reason for requiring linearity of G in necessary-and-sufficient results is that proving the necessary direction is often done via the S-lemma [21, 33], which requires linearity. We go into more detail on this point in Section 2.2. It turns out that for passivity, small gain, circle, and conicity results, linearity is not actually required, even though it is often assumed. Our main result (Theorem 1) provides a unified robust stability result without any linearity requirements.

An earlier version of Theorem 1 [8] used the S-lemma and therefore assumed linearity and was non-constructive.

1.2 Notation

Preliminaries The set $\mathbb{F}$ refers to either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. We use $\mathbb{R}_+$ ($\mathbb{Z}_+$) to denote the nonnegative real numbers (integers). We write the complex conjugate of $x \in \mathbb{F}$ as $\bar{x}$ and the conjugate transpose of a matrix $X \in \mathbb{F}^{m\times n}$ as X^*. We use the symbols $\preceq$, $\prec$, $\succ$, $\succeq$ to denote the (semi)definite partial ordering of matrices in $\mathbb{F}^{n\times n}$. A matrix $A = A^* \in \mathbb{F}^{n\times n}$ is ***indefinite*** if $A \not\preceq 0$ and $A \not\succeq 0$.

Semi-inner products A *semi-inner product space* is a vector space $\mathcal{V}$ over a field $\mathbb{F}$ equipped with a semi-inner product[4] $\langle\cdot,\, \cdot\rangle$, which is an inner product whose associated norm is a seminorm. In other words, $\|x\| := \sqrt{\langle x,\, x\rangle} \geq 0$ for all $x \in \mathcal{V}$, but $\|x\| = 0$ need not imply that $x = 0$.

Relations A *relation* R on $\mathcal{V}$ is a subset of the product space $R \subseteq \mathcal{V}\times\mathcal{V}$. We write $\mathscr{R}(\mathcal{V})$ to denote the set of all relations on $\mathcal{V}$. The ***domain*** of R is defined as $\mathrm{dom}(R) := \{x \in \mathcal{V} \mid (x, y) \in R \text{ for some } y \in \mathcal{V}\}$. For any $x \in \mathrm{dom}(R)$, we write Rx to denote any $y \in \mathcal{V}$ such that $(x, y) \in R$.

We define $\mathcal{V}^2$ to be the augmented vectors $u =: \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right)$ where $u_1, u_2 \in \mathcal{V}$. We overload matrix multiplication in $\mathcal{V}^2$, specifically, for any $\xi, \zeta \in \mathcal{V}^2$ and any matrix $N \in \mathbb{F}^{2\times 2}$,

$$N\xi = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} := \begin{bmatrix} N_{11}\xi_1 + N_{12}\xi_2 \\ N_{21}\xi_1 + N_{22}\xi_2 \end{bmatrix} \in \mathcal{V}^2.$$

Likewise, inner products in $\mathcal{V}^2$ have the interpretation

$$\langle \xi,\, \zeta\rangle = \left\langle \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix},\, \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \right\rangle := \langle \xi_1,\, \zeta_1\rangle + \langle \xi_2,\, \zeta_2\rangle\,.$$

The closed-loop system of Fig. 1 defines the following relations, which characterize pairs of consistent signals.

$$R_{uy} := \left\{(u, y) \in \mathcal{V}^2 \times \mathcal{V}^2 \;\middle|\; (1) \text{ holds for some } e \in \mathcal{V}^2\right\},$$
$$R_{ue} := \left\{(u, e) \in \mathcal{V}^2 \times \mathcal{V}^2 \;\middle|\; (1) \text{ holds for some } y \in \mathcal{V}^2\right\}.$$

[4]We use the convention that a semi-inner product is linear in its second argument, so $\langle x,\, ay + bz\rangle = a\,\langle x,\, y\rangle + b\,\langle x,\, z\rangle$ for all $x, y, z \in \mathcal{V}$ and $a, b \in \mathbb{F}$. Also, $\langle x,\, y\rangle = \overline{\langle y,\, x\rangle}$.

2 Results for semi-inner product spaces

The main result of this section is a robust boundedness theorem defined over a general semi-inner product space. We consider the setup of Fig. 1, where $G \in \mathcal{C}_G$ and $\Phi \in \mathcal{C}_\Phi$ belong to arbitrary sets of (possibly nonlinear) relations.

Theorem 1. *Let $\mathcal{V}$ be a semi-inner product space and let $M = M^* \in \mathbb{F}^{2\times 2}$ be indefinite. Suppose $G \in \mathcal{C}_G$ and consider the three following statements.*

(i) There exists $N = N^* \in \mathbb{F}^{2\times 2}$ satisfying $M + N \prec 0$ such that G satisfies

$$\left\langle \begin{bmatrix} G\xi \\ \xi \end{bmatrix}, N \begin{bmatrix} G\xi \\ \xi \end{bmatrix} \right\rangle \geq 0 \quad \text{for all } \xi \in \operatorname{dom}(G). \tag{3}$$

(ii) There exists $\gamma > 0$ such that for all (u, y, e), if

$$\left\langle \begin{bmatrix} e_2 \\ y_2 \end{bmatrix}, M \begin{bmatrix} e_2 \\ y_2 \end{bmatrix} \right\rangle \geq 0 \tag{4}$$

and (1a), (1c), (1d) *are satisfied, then $\|y\| \leq \gamma \|u\|$.*

(iii) There exists $\gamma > 0$ such that for all $\Phi \in \mathcal{C}_\Phi$, if

$$\left\langle \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix}, M \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix} \right\rangle \geq 0 \quad \text{for all } \xi \in \operatorname{dom}(\Phi), \tag{5}$$

then for all $(u, y) \in R_{uy}$, the following bound holds

$$\|y\| \leq \gamma \|u\|. \tag{6}$$

The following equivalences hold: (i) $\iff$ (ii) $\implies$ (iii).

In Theorem 1, Item (i) is a property of subsystem G, while Item (iii) is a statement about the robust boundedness of the closed loop when G is interconnected with any $\Phi \in \mathcal{C}_\Phi$. The intermediate statement (ii) is similar to (iii), but it concerns robustness with respect to the signals (u, y, e) rather than Φ.

We can view Theorem 1 as a *sufficient* condition for robust boundedness because it proves that (i) $\implies$ (iii). Since Theorem 1 holds for arbitrary choices of $\mathcal{C}_G$ and $\mathcal{C}_\Phi$, it generalizes the results presented in the first half of Table 1. We discuss the details of how to specialize Theorem 1 in Section 3.

We make several remarks about Theorem 1.

Remark 2. *Equation* (6) *can be stated in terms of (u, e) instead of (u, y). Specifically, it is easy to show that* (6) *holds for all $(u, y) \in R_{uy}$ if and only if there exists some $\bar{\gamma} > 0$ such that $\|e\| \leq \bar{\gamma}\|u\|$ holds for all $(u, e) \in R_{ue}$.*

Remark 3. *In Theorem 1, M is assumed to be indefinite. If M is semidefinite instead, it will generally lead to results that are either trivial or vacuous statements, that is, Item (i) and Item (iii) are either always true or always false.*

Remark 4. *In Item (i), we can equivalently replace N by $-M-\varepsilon I$ and modify the statement preceding* (3) *to: "There exists some $\varepsilon > 0$ such that G satisfies* (3)*". We chose the form with M and N to make Theorem 1 more symmetric.*

Remark 5. *Theorem 1 can be further generalized to the case where $G \in \mathcal{C}_G \subseteq \mathcal{R}(\mathcal{V}^n, \mathcal{V}^m)$ (the set of relations on $\mathcal{V}^n \times \mathcal{V}^m$) and $\Phi \in \mathcal{C}_\Phi \subseteq \mathcal{R}(\mathcal{V}^m, \mathcal{V}^n)$. In this case, $M, N \in \mathbb{F}^{(m+n)\times(m+n)}$ would be block 2×2 matrices.*

Remark 6. *In Theorem 1, both G and Φ are relations rather than operators. All relations are invertible, so the closed-loop relations R_{uy} and R_{ue} are always well-defined, but may be* empty*. In such a case,* (6) *is vacuously true. One way to ensure that Theorem 1 is not vacuous is to require the* well-posedness *assumption that R_{uy} (equivalently R_{ue}) is non-empty.*

Since Theorem 1 is expressed in a general semi-inner product space, there need not exist a notion of time and concepts such as causality and stability need not apply. We will see in Section 3 how concepts such as causality, stability, and well-posedness emerge when Theorem 1 is specialized to extended spaces of time-domain signals (Lebesgue spaces) and G and Φ are *operators* rather than relations.

2.1 Proof of sufficiency for Theorem 1

We begin by showing that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) in Theorem 1 for any choice of $\mathcal{C}_G$ and $\mathcal{C}_\Phi$. This proof is similar to [17, Thm. 1]. Pick any (u, y, e) such that (1a), (1c), (1d), and (4) are satisfied. let $\xi = e_1$ in (3). Using (1) to eliminate e_1, e_2, Equations (3) and (4) become:

$$\left\langle \begin{bmatrix} y_1 \\ u_1 + y_2 \end{bmatrix}, N \begin{bmatrix} y_1 \\ u_1 + y_2 \end{bmatrix} \right\rangle \geq 0 \quad \text{and}$$
$$\left\langle \begin{bmatrix} u_2 + y_1 \\ y_2 \end{bmatrix}, M \begin{bmatrix} u_2 + y_1 \\ y_2 \end{bmatrix} \right\rangle \geq 0.$$

Sum the two inequalities above and collect terms to obtain

$$\left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, (M+N) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle + 2 \left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} N_{12} & M_{11} \\ N_{22} & M_{21} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} N_{22} & 0 \\ 0 & M_{11} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\rangle \geq 0.$$

Since $M + N \prec 0$ by assumption, There exists $\eta > 0$ such that $M + N \preceq -\eta I$. Applying this inequality together with Cauchy–Schwarz[5], we obtain

$$-\eta \|y\|^2 + 2r\|y\|\|u\| + q\|u\|^2 \geq 0, \tag{7}$$

where $r := \left\| \left[\begin{smallmatrix} N_{12} & M_{11} \\ N_{22} & M_{21} \end{smallmatrix} \right] \right\|$ and $q := \left\| \left[\begin{smallmatrix} N_{22} & 0 \\ 0 & M_{11} \end{smallmatrix} \right] \right\|$ are standard spectral norms. Dividing by η and completing the square in (7), we obtain $\left(\|y\| - \frac{r}{\eta}\|u\|\right)^2 \leq \frac{r^2+\eta q}{\eta^2}\|u\|^2$, which can be rearranged to establish (ii) with $\gamma = \frac{1}{\eta}\left(r + \sqrt{r^2 + \eta q}\right)$.

To prove (iii), consider some $\Phi \in \mathcal{C}_\Phi$ for which (5) holds. Next, pick $(u, y) \in R_{uy}$ so that there exists (u, y, e) satisfying (1). In particular, (1b) holds, so setting $\xi = e_2$ in (5), we obtain (4) and the rest of the proof is the same as above. ∎

[5] A proof of the Cauchy–Schwarz inequality for general semi-inner product spaces may be found in [6, §1.4].

2.2 Proof of partial necessity for Theorem 1

A popular approach for proving (i) $\Longleftarrow$ (ii) is to use a lossless S-lemma as in [15, Thm. 3] and [20]. However, the S-lemma [21, 33] comes with a drawback: the set of signals (u, y, e) that satisfy the loop equations (1a), (1c), (1d) must be a *subspace*, which requires for example that G be linear. If we assume G is linear, we can prove (i) $\Longleftarrow$ (ii) by adapting the S-lemma for inner product spaces due to Hestenes [11, Thm. 7.1, p. 354] and using a technique similar to that used in [15]. Details of this approach may be found in [7, 8].

It turns out the linearity assumption on G can be dropped entirely if we adopt a different proof approach. To this effect, we will prove the contrapositive $\neg$(i) $\Longrightarrow$ $\neg$(ii) by directly constructing signals (y, u, e) that violate the boundedness condition when (i) fails to hold. Unlike the S-lemma, this approach does not require linearity of G and has the benefit of being constructive, so it produces *worst-case* signals (u, y, e). We state the result in the following lemma.

Lemma 7 (worst-case signals). *Consider the setting of Theorem 1. Suppose that for any N satisfying $M + N \prec 0$, there exists $\xi \in \operatorname{dom}(G)$ such that*

$$\left\langle \begin{bmatrix} G\xi \\ \xi \end{bmatrix}, N \begin{bmatrix} G\xi \\ \xi \end{bmatrix} \right\rangle < 0.$$

Then, for all $\gamma > 0$, there exists (u, y, e) such that:

1. *Equations* (1a), (1c), *and* (1d) *hold.*
2. $\left\langle \begin{bmatrix} e_2 \\ y_2 \end{bmatrix}, M \begin{bmatrix} e_2 \\ y_2 \end{bmatrix} \right\rangle \geq 0.$
3. $\|y\| > \gamma \|u\|$.

Proof. The proof is constructive and explicitly produces the signals (u, y, e) as functions of ξ and γ. See Appendix A.1 for a detailed proof. ∎

Although Theorem 1 is only a sufficient result, the intermediate Item (ii) in Theorem 1 provides an important clue toward finding necessary-and-sufficient results. Specifically, we need only focus on proving (ii) $\Longleftarrow$ (iii) and we can ignore (i) entirely. This is the topic of the next section.

2.3 Toward achieving necessity

Theorem 1 states that for arbitrary choices of $\mathcal{C}_G$ and $\mathcal{C}_\Phi$, the items satisfy the partial equivalence (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii). In general, the three items will not be equivalent. However, there are special choices of $\mathcal{C}_G$ and $\mathcal{C}_\Phi$ that ensure the missing implication (ii) $\Longleftarrow$ (iii) holds and therefore the three items in Theorem 1 become equivalent.

Definition 8. *We say that a pair of constraint sets $(\mathcal{C}_G, \mathcal{C}_\Phi)$* achieves necessity *if such a choice implies that (ii) $\Longleftarrow$ (iii) in Theorem 1.*

Our first observation is that shrinking $\mathcal{C}_G$ or enlarging $\mathcal{C}_\Phi$ preserves the validity of the necessity direction. This leads us to the following proposition.

Proposition 9. *If $(\mathcal{C}_G, \mathcal{C}_\Phi)$ achieves necessity, then $(\mathcal{C}'_G, \mathcal{C}'_\Phi)$ also achieves necessity for any $\mathcal{C}'_G \subseteq \mathcal{C}_G$ and $\mathcal{C}'_\Phi \supseteq \mathcal{C}_\Phi$.*

Proof. If Item (iii) of Theorem 1 holds for $\mathcal{C}'_\Phi$, then it must also hold for $\mathcal{C}_\Phi \subseteq \mathcal{C}'_\Phi$ since the condition ranges over a smaller set of candidate Φ's. Therefore, if $(\mathcal{C}_G, \mathcal{C}_\Phi)$ achieves necessity, then so does $(\mathcal{C}_G, \mathcal{C}'_\Phi)$. Theorem 1 is a statement about $G \in \mathcal{C}_G$, so if $G \in \mathcal{C}'_G \subseteq \mathcal{C}_G$, then Theorem 1 clearly still holds. Thus, if $(\mathcal{C}_G, \mathcal{C}'_\Phi)$ achieves necessity, then so does $(\mathcal{C}'_G, \mathcal{C}'_\Phi)$. This completes the proof. ∎

In light of Proposition 9, our goal should be to find the *smallest* $\mathcal{C}_\Phi$ and *largest* $\mathcal{C}_G$ such that $(\mathcal{C}_G, \mathcal{C}_\Phi)$ achieves necessity. In the remainder of this section, we present two different ways of achieving necessity through specific choices of $\mathcal{C}_G$ and $\mathcal{C}_\Phi$.

Our first result is that Theorem 1 holds if the constraint on $\mathcal{C}_\Phi$ is removed entirely.

Theorem 10 (unconstrained case)**.** *The constraint set $(\mathcal{C}_G, \mathcal{C}_\Phi)$ achieves necessity when $\mathcal{C}_G = \mathcal{C}_\Phi = \mathscr{R}(\mathcal{V})$.*

Proof. We proceed by contradiction. Suppose (ii) fails. Then for all $\gamma > 0$, there exists (u, y, e) satisfying (4) and (1a), (1c), (1d) such that $\|y\| > \gamma \|u\|$. Since $\mathcal{C}_\Phi = \mathscr{R}(\mathcal{V})$, we can pick $\Phi = \{(e_2, y_2)\}$, a singleton relation. Then both (1b) and (5) hold trivially and so (iii) fails, as required. ∎

Theorem 10 may not be particularly satisfying because it requires defining Φ as a singleton relation. A more common use case is when Φ must be defined for all $\mathrm{dom}(\Phi) = \mathcal{V}$.[6] Our second result states that necessity can be achieved by *linear* relations, which we now define.

Definition 11 (linear relation)**.** *Let $\mathcal{V}$ be a semi-inner product space over a field $\mathbb{F}$. Suppose $x_1, x_2, y_1, y_2 \in \mathcal{V}$ and $\alpha_1, \alpha_2 \in \mathbb{F}$. A relation $R \in \mathscr{R}(\mathcal{V})$ is* linear *if for all $(x_1, y_1) \in R$ and $(x_2, y_2) \in R$, we have $(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2) \in R$. We let $\mathscr{L}(\mathcal{V}) \subseteq \mathscr{R}(\mathcal{V})$ denote the set of all linear relations.*

Theorem 12 (linear case)**.** *The constraint set $(\mathcal{C}_G, \mathcal{C}_\Phi)$ achieves necessity when $\mathcal{C}_G = \mathscr{R}(\mathcal{V})$ and $\mathcal{C}_\Phi \supseteq \mathscr{L}(\mathcal{V})$.*

Proof. The proof of Theorem 12 is constructive and explicitly produces a worst-case $\Phi \in \mathscr{L}(\mathcal{V})$. See Appendix A.2 for a detailed proof. ∎

Theorem 10 and 12 both provide conditions that ensure necessity of Theorem 1. In both cases, $\mathcal{C}_G = \mathscr{R}(\mathcal{V})$, so there are no constraints on G; it could be nonlinear, for example.

3 Specialization to extended spaces

In this section, we specialize Theorem 1 to the popular use case where the loop signals (u, y, e) in Fig. 1 are functions of time (either discrete or continuous) and the systems G and Φ are *causal operators* rather than relations. This specialization introduces the familiar concepts of *causality*, *well-posedness*, and *input-output stability.*

This section mirrors Section 2; after some notational preliminaries, we present a corollary to Theorem 1 that holds for causal operators and describe conditions that ensure necessity.

[6] Relations $\Phi \in \mathscr{R}(\mathcal{V})$ that satisfy $\mathrm{dom}(\Phi) = \mathcal{V}$ are known as *serial* or *left-total*. They are also called *multi-valued functions.*

3.1 Notation for extended spaces

In this paper, $\mathcal{L}_2$ denotes a Lebesgue space of functions $\mathcal{T} \to \mathbb{F}^n$. If $\mathcal{T} = \mathbb{Z}_+$, this could be the space $\ell_2^n(\mathbb{F})$ of square-summable sequences with $\langle x,\, y\rangle := \sum_{t=0}^{\infty} x(t)^* y(t)$. Likewise, if $\mathcal{T} = \mathbb{R}_+$, this could be the space $L_2^n(\mathbb{F})$ of square-integrable functions with $\langle x,\, y\rangle := \int_0^\infty x(t)^* y(t)\,\mathrm{d}t$. The definitions that follow apply to commonly used time domains $\mathcal{T}$ such as $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, and $\mathbb{Z}_+$.

For any $T \geq 0$ and function $x : \mathcal{T} \to \mathbb{F}^n$, the ***truncation*** at time T is the function $x_T : \mathcal{T} \to \mathbb{F}^n$ defined as

$$x_T(t) := \begin{cases} x(t) & t \leq T \\ 0 & t > T. \end{cases}$$

The ***extended spaces*** $\mathcal{L}_{2\mathrm{e}} \supseteq \mathcal{L}_2$ are defined as

$$\mathcal{L}_{2\mathrm{e}} := \{x : \mathcal{T} \to \mathbb{F}^n \mid x_T \in \mathcal{L}_2 \text{ for all } T \geq 0\}.$$

We overload the notation $\langle \cdot,\, \cdot\rangle_T$ to denote the truncated semi-inner product on $\mathcal{L}_{2\mathrm{e}}$. That is, $\langle x,\, y\rangle_T := \langle x_T,\, y_T\rangle$. We also define the associated seminorm $\|x\|_T := \sqrt{\langle x,\, x\rangle_T}$.

Definition 13 (causal operators). *A **causal** operator on $\mathcal{L}_{2\mathrm{e}}$ is a function $G : \mathcal{L}_{2\mathrm{e}} \to \mathcal{L}_{2\mathrm{e}}$ with the property that for all $f \in \mathcal{L}_{2\mathrm{e}}$ and $T \geq 0$, we have $(Gf)_T = (Gf_T)_T$. We denote the set of causal operators on $\mathcal{L}_{2\mathrm{e}}$ as $\mathscr{F}(\mathcal{L}_{2\mathrm{e}})$.*

The *shift operator* $S_\tau : \mathcal{L}_{2\mathrm{e}} \to \mathcal{L}_{2\mathrm{e}}$ is defined as $(S_\tau f)(t) = f(t-\tau)$.[7] A causal operator G on $\mathcal{L}_{2\mathrm{e}}$ is said to be ***time-invariant*** if $GS_\tau f = S_\tau G f$ for all $f \in \mathcal{L}_{2\mathrm{e}}$ and all τ. We denote the set of ***linear time-invariant*** (LTI) operators on $\mathcal{L}_{2\mathrm{e}}$ as $\mathscr{L}_{\mathrm{TI}}(\mathcal{L}_{2\mathrm{e}})$.

Given $G \in \mathscr{L}_{\mathrm{TI}}(\mathcal{L}_{2\mathrm{e}})$, we let $\hat{G} : \mathbb{R} \to \mathbb{C}$ denote the ***frequency response*** of G, defined as follows. Let G have transfer function $\mathbf{G}$. If $\mathcal{L}_{2\mathrm{e}} = L_{2\mathrm{e}}$, then $\hat{G}(\omega) := \mathbf{G}(j\omega)$. If $\mathcal{L}_{2\mathrm{e}} = \ell_{2\mathrm{e}}$, then $\hat{G}(\omega) := \mathbf{G}(e^{j\omega})$. We write $\mathcal{L}_\infty$ to denote the set of frequency responses that are essentially bounded for all $\omega \in \mathbb{R}$.

Definition 14 (well-posedness). *If $G, \Phi \in \mathscr{F}(\mathcal{L}_{2\mathrm{e}})$, the interconnection of Fig. 1 is said to be **well-posed** if $R_{uy} \in \mathscr{F}(\mathcal{L}_{2\mathrm{e}})$ (equivalently, $R_{ue} \in \mathscr{F}(\mathcal{L}_{2\mathrm{e}})$). In other words, the closed-loop relation should be a causal operator on $\mathcal{L}_{2\mathrm{e}}$. In this case, we will refer to R_{uy} as the* closed-loop map.

3.2 Main results for extended spaces

Before specializing Theorem 1 to the Lebesgue spaces defined in Section 3.1, we must discuss how well-posedness and causality fit into the picture.

3.2.1 Well-posedness

Assuming G and Φ are relations, as we do in Theorem 1, is not unprecedented in the literature [15, 24, 27, 29, 31, 34]. As mentioned in Remark 6, this strategy ensures well-posedness of any interconnection. However, the closed-loop relations R_{uy} (equivalently R_{ue}) may be empty. When G and Φ are assumed to be operators instead of relations, then well-posedness must either be assumed or proved. Specifically, we need an assurance of the existence and uniqueness of solutions e and y for all choices of u.

[7] By convention, if $\mathcal{L}_{2\mathrm{e}}$ is defined on the time interval $[0,\infty)$, then $f(t) = 0$ for all $t < 0$. So $S_\tau f$ is well-defined for all $f \in \mathcal{L}_{2\mathrm{e}}$ and all τ.

3.2.2 Causality

When working in extended spaces such as $\mathcal{L}_{2e}$, a common assumption is that G and Φ are causal operators [9, 20, 31, 35, 36]. In specializing Theorem 1 to extended spaces, we will let $\mathcal{C}_G$ and $\mathcal{C}_\Phi$ be sets of causal operators on $\mathcal{L}_{2e}$. Then, since a well-posed interconnection of causal maps is causal [29, Prop. 1.2.14], the closed-loop map will be causal.

Although Theorem 1 can in principle be specialized to $\mathcal{V} = \mathcal{L}_2$, this does not yield a fruitful result. In particular, Item (iii) of Theorem 1 would state that for all Φ satisfying the appropriate constraints and for which R_{uy} is non-empty, we would have $\|y\| \le \gamma\|u\|$. Since the typical use case of Theorem 1 is to prove that the closed-loop map $u \mapsto y$ is bounded, this shifts all the burden onto proving R_{uy} is non-empty (well-posedness). In other words, assuming well-posedness amounts to assuming that which we seek to prove.

To resolve the aforementioned problem, we instead specialize Theorem 1 to $\mathcal{V} = \mathcal{L}_{2e}$ and let $\mathcal{C}_G \subseteq \mathscr{F}(\mathcal{L}_{2e})$ and $\mathcal{C}_\Phi \subseteq \mathscr{F}(\mathcal{L}_{2e})$ be arbitrary constraint sets. This leads to the following main result of this section.

Corollary 15 (robust input-output stability on $\mathcal{L}_{2e}$)**.** *Let $M = M^* \in \mathbb{F}^{2\times 2}$ be indefinite. Suppose $G \in \mathcal{C}_G$ and consider the three following statements.*

(i) There exists $N = N^* \in \mathbb{F}^{2\times 2}$ satisfying $M + N \prec 0$ such that for all $\xi \in \mathcal{L}_{2e}$ and $T \ge 0$, G satisfies

$$\left\langle \begin{bmatrix} G\xi \\ \xi \end{bmatrix}, N \begin{bmatrix} G\xi \\ \xi \end{bmatrix} \right\rangle_T \ge 0. \tag{8}$$

(ii) There exists $\gamma > 0$ such that for all (u, y, e), if

$$\left\langle \begin{bmatrix} e_2 \\ y_2 \end{bmatrix}, M \begin{bmatrix} e_2 \\ y_2 \end{bmatrix} \right\rangle_T \ge 0 \quad \text{for all } T \ge 0 \tag{9}$$

and (1a), (1c), (1d) *hold and $u \in \mathcal{L}_2$, then $\|y\| \le \gamma\|u\|$.*

(iii) There exists $\gamma > 0$ such that for all $\Phi \in \mathcal{C}_\Phi$ where the interconnection of Fig. 1 is well-posed, if for all $T \ge 0$ and $\xi \in \mathcal{L}_{2e}$ we have

$$\left\langle \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix}, M \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix} \right\rangle_T \ge 0, \tag{10}$$

then for all $y = R_{uy}u$ with $u \in \mathcal{L}_2$, we have

$$\|y\| \le \gamma\|u\|. \tag{11}$$

The following equivalences hold: (i) $\iff$ (ii) $\implies$ (iii).

Proof. See Appendix B.1. ∎

Corollary 15 is similar to Theorem 1 in that no assumptions are made on $\mathcal{C}_G$ and $\mathcal{C}_\Phi$. The operators G and Φ may be nonlinear or even unbounded. The requirements (8) and (10) involve the truncated T norms, so they are defined even for unbounded signals.[8] Critically, the conclusion (11) states that when $u \in \mathcal{L}_2$, we have $y \in \mathcal{L}_2$, which is a statement about *input-output stability.*

[8] Signals in $\mathcal{L}_{2e}$ may be unbounded, but they cannot have finite escape time.

Corollary 15 is a specialization of Theorem 1 to $\mathcal{V} = \mathcal{L}_{2e}$ and therefore it is in the spirit of other time-domain results such as the conic sector theorem [34], and extended conic sector theorem [2].

As with general semi-inner product spaces, we can inquire about conditions on $\mathcal{C}_G$ and $\mathcal{C}_\Phi$ for which (ii) $\Longleftarrow$ (iii) and the result becomes necessary and sufficient for input-output stability. This is shown in Theorem 18.

Definition 16. *We say that a pair of constraint sets* $(\mathcal{C}_G, \mathcal{C}_\Phi)$ achieves necessity on $\mathcal{L}_{2e}$ *if such a choice implies that (ii)* $\Longleftarrow$ *(iii) in Corollary 15.*

As in the semi-inner product setting, shrinking $\mathcal{C}_G$ or enlarging $\mathcal{C}_\Phi$ preserves the validity of the necessity direction. So Proposition 9 also holds on $\mathcal{L}_{2e}$ when $\mathcal{C}_G$ and $\mathcal{C}_\Phi$ are subsets of $\mathscr{F}(\mathcal{L}_{2e})$, the causal operators on $\mathcal{L}_{2e}$.

Finding constraint sets that achieve necessity on $\mathcal{L}_{2e}$ is more challenging than in the general semi-inner product setting because we require Φ to be a causal operator. Indeed, Theorem 10 constructs a worst-case Φ using a singleton relation, which is not a valid operator. Likewise, Theorem 12 constructs a linear worst-case Φ from the worst-case signals (e_2, y_2), but the resulting Φ is generally not causal.

One way to eliminate these difficulties is to make further assumptions on G. Specifically, we will assume G is LTI, which allows for the following equivalence between inner products and frequency domain inequalities (FDIs).

Lemma 17. *Let* $N = N^* \in \mathbb{F}^{2\times 2}$ *be indefinite and let* $P \in \mathbb{F}^{2\times 2}$ *diagonalize* N*, with* $N = P^* \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] P$*. Suppose* $G \in \mathscr{L}_{\mathrm{TI}}(\mathcal{L}_{2e})$ *with frequency response* $\hat{G}$*. The following two statements are equivalent.*

1. *For all* $T \geq 0$ *and for all* $\xi \in \mathcal{L}_{2e}$*, we have*
$$\left\langle \begin{bmatrix} G\xi \\ \xi \end{bmatrix}, N \begin{bmatrix} G\xi \\ \xi \end{bmatrix} \right\rangle_T \geq 0. \tag{12}$$
2. *The following frequency-domain inequality (FDI) holds:*
$$\begin{bmatrix} \hat{G}(\omega) \\ 1 \end{bmatrix}^* N \begin{bmatrix} \hat{G}(\omega) \\ 1 \end{bmatrix} \geq 0 \quad \text{for almost all } \omega \in \mathbb{R} \tag{13}$$
and $(P_{11}G + P_{12})(P_{21}G + P_{22})^{-1}$ *maps* $\mathcal{L}_2 \to \mathcal{L}_2$*.*

Proof. See Appendix B.2 for a detailed proof. ∎

We can now prove a version of Theorem 12 when G is constrained to be LTI.

Theorem 18 (LTI case)**.** *The constraint set* $(\mathcal{C}_G, \mathcal{C}_\Phi)$ *achieves necessity on* $\mathcal{L}_{2e}$ *when* $\mathcal{C}_G \subseteq \mathscr{L}_{\mathrm{TI}}(\mathcal{L}_{2e}) \subseteq \mathcal{C}_\Phi$*.*

Proof. The proof of Theorem 18 is constructive and inspired by Vidyasagar's necessary and sufficient circle criterion [31, Thm. 6.6.126]. See Appendix B.3 for the proof. ∎

Remark 19. *As noted in Section 2.2, if we assume that* G *is linear, the S-lemma can be used to prove the implication (i)* $\Longleftarrow$ *(ii) in Corollary 15. However, this implication still holds even when* G *is nonlinear. Theorem 18 also assumes that* G *is LTI (in the* $\mathcal{L}_{2e}$ *setting), but instead proves the implication (ii)* $\Longleftarrow$ *(iii), for which the S-lemma cannot be used. Linearity is only used the* $\mathcal{L}_{2e}$ *setting to deal with the requirement that* Φ *be causal. In the semi-inner product setting, linearity of* G *is not required (see Theorems 10 and 12).*

We can combine Theorem 18 with Corollary 15 and Lemma 17 to obtain a necessary and sufficient condition relating frequency-domain properties of G with robust closed-loop stability. We diagonalize M instead of N for convenience, but the result can be formulated equivalently either way.

Theorem 20 (LTI case in the frequency domain)**.** *Let $M = M^* \in \mathbb{F}^{2\times 2}$ be indefinite and let $P \in \mathbb{F}^{2\times 2}$ diagonalize M with $M = P^* \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] P$. Suppose $G \in \mathscr{L}_{\mathrm{TI}}(\mathcal{L}_{2\mathrm{e}})$ has frequency response $\hat{G}$ and let $\mathcal{C}_\Phi \supseteq \mathscr{L}_{\mathrm{TI}}(\mathcal{L}_{2\mathrm{e}})$. The following two statements are equivalent.*

(i) There exists $N = N^* \in \mathbb{F}^{2\times 2}$ satisfying $M + N \prec 0$ such that the following FDI holds for almost all $\omega \in \mathbb{R}$:

$$\begin{bmatrix} \hat{G}(\omega) \\ 1 \end{bmatrix}^* N \begin{bmatrix} \hat{G}(\omega) \\ 1 \end{bmatrix} \geq 0, \tag{14}$$

and $(P_{11}G + P_{12})(P_{21}G + P_{22})^{-1}$ maps $\mathcal{L}_2 \to \mathcal{L}_2$.

(ii) There exists $\gamma > 0$ such that for all $\Phi \in \mathcal{C}_\Phi$ where the interconnection of Fig. 1 is well-posed, if for all $T \geq 0$ and $\xi \in \mathcal{L}_{2\mathrm{e}}$ we have

$$\left\langle \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix}, M \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix} \right\rangle_T \geq 0, \tag{15}$$

then for all $y = R_{uy}u$ with $u \in \mathcal{L}_2$, we have

$$\|y\| \leq \gamma \|u\|. \tag{16}$$

When M and N are suitably chosen in Theorem 20, the frequency domain condition (14) can take on a familiar form, such as the classical circle criterion. In Section 4, we describe how different choices of M and N can be used to recover existing results in the literature.

The condition that $(P_{11}G + P_{12})(P_{21}G + P_{22})^{-1}$ maps $\mathcal{L}_2 \to \mathcal{L}_2$ is equivalent to $(P_{11}\mathbf{G} + P_{12})(P_{21}\mathbf{G}+P_{22})^{-1}$ being stable, where $\mathbf{G}$ is the transfer function of G. If we only have access to the frequency response $\hat{G}$, then the condition can be verified using the Nyquist criterion instead. Several existing results in the literature are expressed in this way, see for example [9, Thm. V.2.10], [13, Thm. 7.2], and [31, Thm. 6.6.126].

The frequency-domain condition (14) can be verified graphically, or if we have a state space representation for G, we can use the Kalman–Yakubovich–Popov (KYP) lemma to transform (14) into an equivalent linear matrix inequality (LMI), which admits a numerically tractable solution. We state the KYP lemma here (in the general MIMO case) for completeness.

Lemma 21. *Let $N = N^\mathsf{T} \in \mathbb{R}^{(p+m)\times(p+m)}$ be given and suppose $G \in \mathscr{L}_{\mathrm{TI}}(\mathcal{L}_{2\mathrm{e}}^m \to \mathcal{L}_{2\mathrm{e}}^p)$ is a finite-dimensional system with frequency response $\hat{G} \in \mathcal{L}_\infty^{p\times m}$. The following statements are equivalent.*

1. *The following frequency-domain inequality (FDI) holds for almost all $\omega \in \mathbb{R}$.*

$$\begin{bmatrix} \hat{G}(\omega) \\ I \end{bmatrix}^* N \begin{bmatrix} \hat{G}(\omega) \\ I \end{bmatrix} \succeq 0. \tag{17}$$

2. *Suppose G has a minimal realization (A, B, C, D). Then the following linear matrix inequality (LMI) has a solution $P = P^\mathsf{T}$. In the case $\mathcal{L}_2 = L_2$ (continuous time),*

$$\begin{bmatrix} A^\mathsf{T}P + PA & PB \\ B^\mathsf{T}P & 0 \end{bmatrix} \preceq \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\mathsf{T} N \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}, \tag{18}$$

and in the case $\mathcal{L}_2 = \ell_2$ (discrete time),

$$\begin{bmatrix} A^\mathsf{T} P A - P & A^\mathsf{T} P B \\ B^\mathsf{T} P A & B^\mathsf{T} P B \end{bmatrix} \preceq \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\mathsf{T} N \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}. \tag{19}$$

Proof. See for example [23]. ∎

Remark 22. *Standard Lyapunov arguments may be applied to further refine Lemma 21. For example, If $N_{11} \preceq 0$ and G is stable, then we have $P \succeq 0$ in* (18) *or* (19)*.*

Table 2: Sufficient conditions for stability drawn from the literature. For each result, we show the choices of M and N and the condition $M + N \prec 0$ that allows Corollary 15 to recover the result. Under the assumptions of Theorem 18 these conditions become necessary as well, as described in Section 4. We use a positive feedback convention as in Fig. 1. To use the negative feedback convention instead, replace N by $\tilde{N}$ as described in Section 4.1.2.

Name of Theorem	$\boldsymbol{M}$	$\boldsymbol{N}$	$\boldsymbol{M + N \prec 0}$
Conic sector theorem [34, Thm. 2a, all three cases] or [2, Thm. 3.1, all parts of Case 1]	$\begin{bmatrix} \frac{-(a+\Delta)(b-\Delta)}{b-a-2\Delta} & \frac{-a-b}{2(b-a-2\Delta)} \\ \frac{-a-b}{2(b-a-2\Delta)} & \frac{-1}{b-a-2\Delta} \end{bmatrix}$	$\begin{bmatrix} \frac{ab}{b-a+2ab\delta} & \frac{a+b}{2(b-a+2ab\delta)} \\ \frac{a+b}{2(b-a+2ab\delta)} & \frac{(1+a\delta)(1-b\delta)}{b-a+2ab\delta} \end{bmatrix}$	$a < b$, and either $\delta = 0, \Delta > 0$ or $\delta > 0, \Delta = 0$.
Extended conic sector theorem [2, Thm. 3.1, all parts of Case 2]	$\begin{bmatrix} \frac{(a-\Delta)(b+\Delta)}{b-a+2\Delta} & \frac{a+b}{2(b-a+2\Delta)} \\ \frac{a+b}{2(b-a+2\Delta)} & \frac{1}{b-a+2\Delta} \end{bmatrix}$	$\begin{bmatrix} \frac{-ab}{b-a-2ab\delta} & \frac{-a-b}{2(b-a-2ab\delta)} \\ \frac{-a-b}{2(b-a-2ab\delta)} & \frac{-(1-a\delta)(1+b\delta)}{b-a-2ab\delta} \end{bmatrix}$	Same as above.
Extended passivity [31, Thm. 6.6.58]	$\begin{bmatrix} -\varepsilon_2 & \frac{1}{2} \\ \frac{1}{2} & -\delta_2 \end{bmatrix}$	$\begin{bmatrix} -\delta_1 & -\frac{1}{2} \\ -\frac{1}{2} & -\varepsilon_1 \end{bmatrix}$	$\delta_1 + \varepsilon_2 > 0$ and $\delta_2 + \varepsilon_1 > 0$.
Small gain theorem [13, Thm. 5.6]	$\begin{bmatrix} \gamma_2 & 0 \\ 0 & -1/\gamma_2 \end{bmatrix}$	$\begin{bmatrix} -1/\gamma_1 & 0 \\ 0 & \gamma_1 \end{bmatrix}$	$\gamma_1 \gamma_2 < 1$.

4 Recovering existing results

Our results of Sections 2 and 3 can be used to recover a variety of existing robust stability results from the literature. To demonstrate this versatility, we begin with *necessity-preserving specializations.* These are choices that do not affect any of our proofs and thus yield equally strong results.

4.1 Necessity-preserving specializations

4.1.1 Different spaces

The most common spaces are $\mathcal{L}_{2\mathrm{e}} = L_{2\mathrm{e}}$ and $\mathcal{L}_{2\mathrm{e}} = \ell_{2\mathrm{e}}$ (continuous or discrete time, respectively). The results are essentially identical in these two cases. We saw in the proof of Theorem 18 that it is also possible to apply Theorem 1 directly to a space $\mathcal{L}_\infty$ of frequency responses.

4.1.2 Sign conventions

Although we used the positive feedback sign convention in Fig. 1, using the negative feedback convention instead simply amounts to replacing N by $\tilde{N}$ in Theorem 12, Corollary 15, and Theorem 20, where

$$N := \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \quad \text{and} \quad \tilde{N} := \begin{bmatrix} N_{11} & -N_{12} \\ -N_{21} & N_{22} \end{bmatrix}.$$

Alternatively, we could replace M by $\tilde{M}$ (multiply the off-diagonal elements of M by -1). These changes amount to replacing G by $-G$ or Φ by $-\Phi$, respectively.

4.1.3 Strictness of inequalities

Robust stability results generally involve non-intersecting sets (typically cones), so for the inequalities describing the admissible systems G and Φ, one will typically be strict and the other will be nonstrict. Our symmetric formulations using a coupling constraint $M + N \prec 0$ avoids the need to associate strictness with either G or Φ; both cases can be represented by suitable choice of M and N.

4.1.4 Different cones

Different choices of the matrices M and N in our results allow the representation of different cones. For example, we can represent different flavors of passivity (input-strict passivity, output-strict passivity, extended passivity), small-gain results, the circle criterion, and other conic sectors that allow G or Φ to be unbounded/unstable.

To illustrate these various transformations, consider for example the classical passivity result by Vidyasagar, which is a sufficient-only result, and may be found in [31, Thm. 6.7.43].

Theorem 23 (Vidyasagar)**.** *Consider the system*

$$\begin{cases} e_1 = u_1 - y_2, & y_1 = Ge_1 \\ e_2 = u_2 + y_1, & y_2 = \Phi e_2 \end{cases}$$

Suppose there exist constants ε_1, ε_2, δ_1, δ_2 such that for all $\xi \in \ell_{2\mathrm{e}}$ and for all $T \geq 0$

$$\langle \xi, G\xi \rangle_T \geq \varepsilon_1 \|\xi\|_T^2 + \delta_1 \|G\xi\|_T^2, \tag{20a}$$

$$\langle \xi, \Phi\xi \rangle_T \geq \varepsilon_2 \|\xi\|_T^2 + \delta_2 \|\Phi\xi\|_T^2. \tag{20b}$$

Then the system is ℓ_2-stable if $\delta_1 + \varepsilon_2 > 0$ and $\delta_2 + \varepsilon_1 > 0$.

Theorem 23 uses a negative sign convention and is expressed in discrete time. To obtain a corresponding sufficient result, apply Corollary 15 with $\mathcal{L}_{2\mathrm{e}} = \ell_{2\mathrm{e}}$. Comparing (8) and (10) to (20), which yields the following $\tilde{N}$, N, and M.

$$\tilde{N} = \begin{bmatrix} -\delta_1 & \frac{1}{2} \\ \frac{1}{2} & -\varepsilon_1 \end{bmatrix}, \; N = \begin{bmatrix} -\delta_1 & -\frac{1}{2} \\ -\frac{1}{2} & -\varepsilon_1 \end{bmatrix}, \; M = \begin{bmatrix} -\varepsilon_2 & \frac{1}{2} \\ \frac{1}{2} & -\delta_2 \end{bmatrix}.$$

In Corollary 15, we require $M + N \prec 0$; thus $\delta_1 + \varepsilon_2 > 0$ and $\delta_2 + \varepsilon_1 > 0$, which recovers Theorem 23.

In Theorem 23 (and in Corollary 15), the systems G and Φ need not be linear. If we want a necessary version of this passivity result, we can assume $\mathcal{C}_G \subseteq \mathscr{L}_{\mathrm{TI}}(\mathcal{L}_{2\mathrm{e}}) \subseteq \mathcal{C}_\Phi$ and apply Theorem 18. In other words, we have necessity if G is LTI and we are certifying robustness with respect to a class $\mathcal{C}_\Phi$ that includes all LTI systems. This last requirement means that Φ may have *memory*.

Corollary 15 can also be specialized to recover the small-gain theorem [13, Thm. 5.6], extended conic sector theorem [2], circle criterion [12], and other versions of passivity such as Vidyasagar [31, Thm. 6.6.58] and Khong & van der Schaft [15]. We summarize these results in Table 2, along with the appropriate choice of M, N, and the associated condition $M + N \prec 0$.

4.2 Recovering sufficient-only results

Many robust stability results in the literature (see Tables 1 and 2) are presented as being sufficient but *not* necessary. For those results involving *static multipliers* (passivity, small gain, conicity, etc.), we will argue that sufficiency follows from Theorem 1 or Corollary 15, while the lack of necessity follows from the necessary conditions described in Theorems 10, 18, and 20 not being met. In other words, we can view these sufficient results as direct consequences of their necessary-and-sufficient counterparts. We now discuss three ways in which this phenomenon can manifest itself in practice.

4.2.1 Alternative inner products

We showed that in Theorem 1, the partial converse (i) $\Longleftarrow$ (ii) holds, and this fact is independent of the choice of semi-inner product. However, when we specialized to $\mathcal{L}_{2\mathrm{e}}$ in Section 3, we sought to prove input-output stability. Proving stability requires showing that certain properties of $\mathcal{L}_{2\mathrm{e}}$ signals imply boundedness in some other norm (e.g., the $\mathcal{L}_2$ norm). Our ability to do this *does* depend on the choice of semi-inner product used.

As an example, consider the space $\mathcal{V} = \mathcal{L}_{2\mathrm{e}}$ and define the *pointwise* semi-inner product as $\langle x,\, y\rangle_{p,T} := x(T)^* y(T)$. This may arise, for example, if G and Φ satisfy a stronger notion of *pointwise passivity* rather than the standard definition using $\langle\cdot,\,\cdot\rangle_T$. In this case, we can prove a result very similar to Corollary 15, except we can only prove the implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). The partial converse (i) $\Longleftarrow$ (ii) does not hold because having $\|y\| \le \gamma\|u\|$ for all $u \in \mathcal{L}_2$ *does not* imply that $|y(T)| \le \gamma|u(T)|$ for all $T \ge 0$.

4.2.2 Relaxed constraints on subsystems

A straightforward way to relax a necessary-and-sufficient result is to relax the conic constraints that characterize G and Φ. For example, consider Theorem 20. In Item (i), replace the condition $M + N \prec 0$ by $M + N \prec -\eta I$ for some $\eta > 0$. Equivalently, replace N by $\hat{N} \preceq \lambda N$ for some $\lambda > 0$ in (14). Naming the new condition (i′), we have that (i′) $\Longrightarrow$ (i). Likewise, define (ii′) to be the same as (ii) except M is replaced by some $\hat{M} \preceq \mu M$ for some $\mu > 0$ in (15). Then, we have that (ii) $\Longrightarrow$ (ii′). Putting these facts together, we obtain the implications: (i′) $\Longrightarrow$ (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (ii′). The implication (i′) $\Longrightarrow$ (ii′) cannot be reversed in general, so the necessary-and-sufficient condition has become sufficient-only.

Geometrically, constraints such as $\hat{M} \preceq \lambda M$ correspond to nested cones (the S-lemma). For example, consider the sets:

$$S := \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} x \\ y \end{bmatrix}^\mathsf{T} M \begin{bmatrix} x \\ y \end{bmatrix} \ge 0 \right\} \qquad \text{with } M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\hat{S} := \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} x \\ y \end{bmatrix}^\mathsf{T} \hat{M} \begin{bmatrix} x \\ y \end{bmatrix} \ge 0 \right\} \qquad \text{with } \hat{M} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

The set S is the conic region between $y = -x$ and $y = x$, and $\hat{S}$ is the conic region between $y = 0$ and $y = x$. We have $\hat{S} \subseteq S$ and also $\hat{M} \preceq M$. Importantly, the implication is reversed when considering robust constraint satisfaction such as in (ii). For example, given some property $\mathbb{P}$, we have:

$$\mathbb{P} \text{ holds for all } \Phi \in S \implies \mathbb{P} \text{ holds for all } \Phi \in \hat{S}.$$

We can also relax the constraints on G and Φ by applying Proposition 9. Namely, if we use $\mathcal{C}'_G \supseteq \mathcal{C}_G$ and $\mathcal{C}'_\Phi \subseteq \mathcal{C}_\Phi$ in any of our results, we will introduce conservatism.

4.2.3 Memoryless nonlinearities

As mentioned in the previous item, choosing $\mathcal{C}'_\Phi \subseteq \mathcal{C}_\Phi$ will generally introduce conservatism. Perhaps the most famous such constraint is to restrict $\Phi : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ to be *memoryless*, which means that Φ operates pointwise in time. In this case, we can write $(\Phi x)(t) = \phi_t(x(t))$ for some functions ϕ_t. Note that when Φ is memoryless, we have:

$$\left\langle \begin{bmatrix} x \\ \Phi x \end{bmatrix}, M \begin{bmatrix} x \\ \Phi x \end{bmatrix} \right\rangle_T \geq 0 \quad \text{for all } x \in \mathcal{L}_{2e}, T \geq 0$$
$$\iff \left\langle \begin{bmatrix} x \\ \Phi x \end{bmatrix}, M \begin{bmatrix} x \\ \Phi x \end{bmatrix} \right\rangle_{p,T} \geq 0 \quad \text{for all } x \in \mathcal{L}_{2e}, T \geq 0.$$

where $\langle \cdot, \cdot \rangle_{p,T}$ is the pointwise inner product defined in Section 4.2.1. Indeed, the classical circle criterion is typically stated with a memoryless or static[9] nonlinearity Φ and a pointwise inner product characterizing the conic constraint. Finding a necessary and sufficient condition for robust stability under these assumptions remains an open problem [19].

Brockett's counterexample: In [4], a counterexample was presented that showed the standard circle criterion is only sufficient when Φ is memoryless. The closed-loop system for this example is described by the differential equation

$$\ddot{y} + 2\dot{y} + f(t)y = 0, \tag{21}$$

where $f(t)$ satisfies $0 \leq f(t) \leq k$ for all t. In the context of Theorem 15, G is described by the transfer function $\frac{-1}{s(s+2)}$ and Φ is linear and memoryless, described by $(\Phi y)(t) = f(t)y(t)$.

The constraint on f corresponds to using $M = \left[\begin{smallmatrix} 0 & k \\ k & -2 \end{smallmatrix}\right]$. Applying Corollary 15, we seek to satisfy Item (i). Using $N = -M - \varepsilon I$ and applying Lemma 17, this amounts to certifying that for all $\omega \in \mathbb{R}$, we have

$$\begin{bmatrix} \hat{G}(\omega) \\ 1 \end{bmatrix}^* M \begin{bmatrix} \hat{G}(\omega) \\ 1 \end{bmatrix} < 0, \quad \text{where } \hat{G}(\omega) = \frac{-1}{j\omega(j\omega + 2)}.$$

This condition simplifies to $k < 4 + \omega^2$. Therefore, we conclude that we have robust stability of the closed-loop map (21) whenever $0 \leq k \leq 4$. It is explained in [4] that robust stability actually holds for $0 \leq k \leq 11.6$, therefore the circle criterion is sufficient-only.

The example above satisfies all the conditions of Theorem 20 except that $\mathcal{C}_\Phi \not\supseteq \mathscr{L}_{\mathrm{TI}}(\mathcal{L}_{2e})$, since Φ is required to be memoryless and LTI systems are not memoryless in general.

If we allow Φ to be LTI and assume $k = 4 + \varepsilon$ for any $\varepsilon > 0$, then the frequency-domain condition is violated for any $0 < \omega_0 < \sqrt{\varepsilon}$. Following the construction described in the proof of Theorem 18, we can construct a Φ that is a static gain cascaded with a pure delay that depends on ω_0 and achieves arbitrarily large input-output gain. Such a Φ is *not* memoryless.

Therefore, the condition $0 \leq k \leq 4$ does render the circle criterion necessary and sufficient if Φ can have memory.

[9] The map $\Phi : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is called *static* if it is both memoryless and time-invariant. A static Φ satisfies $(\Phi x)(t) = \phi(x(t))$ for some function ϕ.

5 Conclusion

We studied robust stability results involving a plant G connected with a nonlinearity Φ belonging to a conic sector, e.g. passivity, small-gain, circle criterion, conicity, or extended conicity. Our goal was to distill the vast literature on this topic and state the most general and unified results possible.

Robust boundedness results are often stated in the form of *sufficient conditions*. Our first observation is that assumptions made in these results can always be relaxed in such a way that the sufficient conditions become necessary as well.

We distinguish between two types of necessity that are often confounded in the literature. In particular, when the sufficient conditions for robust boundedness are not met, we may seek:

1. *Worst-case signals* that yield an unbounded closed loop. This form of necessity *always* holds, even for nonlinear plants (Theorem 1 and Corollary 15).

2. *A worst-case nonlinearity* Φ that yields an unbounded closed loop. This form of necessity *always* holds in the general semi-inner product setting, and we show how to construct a *linear* worst-case Φ (Theorem 12). In the extended ($\mathcal{L}_{2\mathrm{e}}$) setting where G and Φ are *causal operators*, we show how to construct a linear worst-case Φ when the plant is LTI (Theorem 18). Our constructed Φ consists of a static gain cascaded with a pure time delay.

Looking beyond the scope of this paper, it would be interesting to see if our semi-inner product framework could be used to recover results involving dynamic constraints (dissipativity, multiplier theory, integral quadratic constraints).

Our work also delineates (see Section 4) the ways in which necessity may be lost. This points to areas where the existing sufficient conditions could potentially be improved. Of particular note are problems where alternative inner products are used, or problems where the nonlinearity is memoryless.

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A Proofs for semi-inner product spaces

This section contains proofs related to the partial necessity of Theorem 1 (Lemma 7) and the full necessity for the case where $\mathcal{C}_\Phi$ contains all linear relations (Theorem 12).

A.1 Proof of Lemma 7

Proof. Let $\gamma > 0$ be arbitrary. Here are the steps to the construction.

1. Since M is indefinite, we can write $M = P^* \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] P$ for some invertible $P \in \mathbb{F}^{2\times 2}$.

2. Define: $\tilde{\gamma} := \underline{\sigma}(P)^{-1} \left(\gamma\, \overline{\sigma}(P) + \overline{\sigma}\left(\left[\begin{smallmatrix} P_{12} & 0 \\ 0 & P_{21} \end{smallmatrix}\right]\right)\right)$, where $\overline{\sigma}(\cdot)$ and $\underline{\sigma}(\cdot)$ denote the maximum and minimum singular value, respectively.

3. Choose any $\delta \in \mathbb{R}$ such that $0 < \delta < \frac{1}{\tilde{\gamma}+1}$. Rearranging this inequality, we obtain $\tilde{\gamma} < \frac{1-\delta}{\delta} < \frac{1}{\delta}$. Therefore,
$$(1-\delta)^2 - \tilde{\gamma}^2\delta^2 > 0 \quad \text{and} \quad 1 - \tilde{\gamma}^2\delta^2 > 0. \tag{22}$$

4. Choose any $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < \frac{2\delta}{1+\delta^2}$. Rearranging this inequality, we obtain
$$\sqrt{\frac{1+\varepsilon}{1-\varepsilon}} < \frac{1+\delta}{1-\delta}. \tag{23}$$

5. Let $N = -M - \varepsilon P^* P$ and let ξ be such that
$$\left\langle \begin{bmatrix} G\xi \\ \xi \end{bmatrix}, N \begin{bmatrix} G\xi \\ \xi \end{bmatrix} \right\rangle < 0. \tag{24}$$

6. Define the signals $(\tilde{u}, \tilde{y}, \tilde{e})$ as follows:
$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{e}_1 \end{bmatrix} := P \begin{bmatrix} G\xi \\ \xi \end{bmatrix}, \tag{25}$$
$$\begin{bmatrix} \tilde{e}_2 \\ \tilde{y}_2 \end{bmatrix} := \begin{bmatrix} 1+\delta & 0 \\ 0 & 1-\delta \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{e}_1 \end{bmatrix}, \tag{26}$$
$$\begin{bmatrix} -\tilde{u}_2 \\ \tilde{u}_1 \end{bmatrix} := \begin{bmatrix} -\delta & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{e}_1 \end{bmatrix}. \tag{27}$$

7. Finally, define the transformed signals:
$$\begin{bmatrix} y_1 \\ e_1 \end{bmatrix} := P^{-1} \begin{bmatrix} \tilde{y}_1 \\ \tilde{e}_1 \end{bmatrix} = \begin{bmatrix} G\xi \\ \xi \end{bmatrix}, \tag{28}$$
$$\begin{bmatrix} e_2 \\ y_2 \end{bmatrix} := P^{-1} \begin{bmatrix} \tilde{e}_2 \\ \tilde{y}_2 \end{bmatrix}, \tag{29}$$
$$\begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} := P^{-1} \begin{bmatrix} -\tilde{u}_2 \\ \tilde{u}_1 \end{bmatrix}. \tag{30}$$

Steps 1–7 provide the construction of (u, y, e). We now verify that this choice satisfies Items 1–3 of Lemma 7.

By adding and subtracting equations above, we obtain (25) = (26) + (27), and therefore (28) = (29) + (30). This immediately verifies that (1a), (1c), and (1d) are satisfied.

Based on how ξ is defined in (24), and substituting the choice of N from Step 5 and the factorization for M from Step 1, we obtain $(-1-\varepsilon)\|\tilde{y}_1\|^2 + (1-\varepsilon)\|\tilde{e}_1\|^2 < 0$. From this inequality, it is clear that $\|\tilde{y}_1\| \neq 0$. So we conclude that
$$\frac{\|\tilde{e}_1\|}{\|\tilde{y}_1\|} < \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}. \tag{31}$$

Combining (31) and (23), we obtain $\frac{\|\tilde{e}_1\|}{\|\tilde{y}_1\|} < \frac{1+\delta}{1-\delta}$. Rearranging, we obtain $(1-\delta)\|\tilde{e}_1\| < (1+\delta)\|\tilde{y}_1\|$, which based on our definitions in Step 6, is equivalent to $\|\tilde{y}_2\| < \|\tilde{e}_2\|$. Rewriting as a quadratic form and converting coordinates, we can invoke the definitions in Steps 1 and 7 to obtain:
$$\left\langle \begin{bmatrix} \tilde{e}_2 \\ \tilde{y}_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{e}_2 \\ \tilde{y}_2 \end{bmatrix} \right\rangle > 0 \iff \left\langle \begin{bmatrix} e_2 \\ y_2 \end{bmatrix}, M \begin{bmatrix} e_2 \\ y_2 \end{bmatrix} \right\rangle > 0.$$

This verifies Item 2 of Lemma 7. Based on the inequalities in (22) and the fact that $\|\tilde{y}_1\| > 0$ derived above, we have

$$\left(1 - \tilde{\gamma}^2\delta^2\right) \|\tilde{y}_1\|^2 + \left((1-\delta)^2 - \tilde{\gamma}^2\delta^2\right) \|\tilde{e}_1\|^2 > 0. \tag{32}$$

Applying the definitions from (26) and (27), Equation (32) is equivalent to: $\|\tilde{y}_1\|^2 + \|\tilde{y}_2\|^2 > \tilde{\gamma}^2 \left(\|\tilde{u}_1\|^2 + \|\tilde{u}_2\|^2\right)$, or more compactly, $\|\tilde{y}\| > \tilde{\gamma}\|\tilde{u}\|$. We now apply the definitions (28)–(30) and the closed-loop equations (1a), (1c), (1d) to obtain a bound in terms of (u, y, e). For the upper bound,

$$\begin{aligned}
\|\tilde{y}\| &= \left\| \begin{bmatrix} P_{11}y_1 + P_{12}e_1 \\ P_{21}e_2 + P_{22}y_2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} P_{11}y_1 + P_{12}(y_2 + u_1) \\ P_{21}(y_1 + u_2) + P_{22}y_2 \end{bmatrix} \right\| \\
&= \left\| P \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} P_{12} & 0 \\ 0 & P_{21} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\| \\
&\le \overline{\sigma}(P)\|y\| + \overline{\sigma}\left(\left[\begin{smallmatrix} P_{12} & 0 \\ 0 & P_{21} \end{smallmatrix}\right]\right) \|u\|.
\end{aligned}$$

For the lower bound,

$$\tilde{\gamma}\|\tilde{u}\| = \tilde{\gamma} \left\| P \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} \right\| \ge \tilde{\gamma}\, \underline{\sigma}(P)\|u\|.$$

Combining the upper and lower bounds, we obtain:

$$\tilde{\gamma}\, \underline{\sigma}(P)\|u\| < \overline{\sigma}(P)\|y\| + \overline{\sigma}\left(\left[\begin{smallmatrix} P_{12} & 0 \\ 0 & P_{21} \end{smallmatrix}\right]\right) \|u\|.$$

Rearranging, we obtain

$$\underbrace{\overline{\sigma}(P)^{-1}\Big(\tilde{\gamma}\, \underline{\sigma}(P) - \overline{\sigma}\left(\left[\begin{smallmatrix} P_{12} & 0 \\ 0 & P_{21} \end{smallmatrix}\right]\right)\Big)}_{\gamma}\|u\| < \|y\|.$$

Based on how we defined $\tilde{\gamma}$ in Step 2, the above simplifies to $\|y\| > \gamma\|u\|$, which proves Item 3 from Lemma 7. ∎

A.2 Proof of Theorem 12

We begin by proving the following lemma, which states that a single pair of points satisfying a quadratic constraint can be extended to a linear relation that satisfies the quadratic constraint everywhere.

Lemma 24 (extension lemma)**.** *Let $\mathcal{V}$ be a semi-inner product space and let $M = M^* \in \mathbb{F}^{2\times 2}$ be indefinite. Suppose $e, y \in \mathcal{V}$ satisfy the inequality*

$$\left\langle \begin{bmatrix} e \\ y \end{bmatrix}, M \begin{bmatrix} e \\ y \end{bmatrix} \right\rangle \ge 0.$$

There exists $\Phi \in \mathscr{L}(\mathcal{V})$ such that:

1. *$(e, y) \in \Phi$.*
2. *$\left\langle \begin{bmatrix} x \\ \Phi x \end{bmatrix}, M \begin{bmatrix} x \\ \Phi x \end{bmatrix} \right\rangle \ge 0$ for all $x \in \mathrm{dom}(\Phi)$.*

Moreover, if $\|e\| > 0$, we can construct Φ that is a linear function*, with* $\mathrm{dom}(\Phi) = \mathcal{V}$*.*

Using Lemma 24, we can prove Theorem 12 by contradiction. Indeed, if Item (ii) of Theorem 1 fails, then for any $\gamma > 0$, there exist $e_2, y_2 \in \mathcal{V}$ such that (4) and (1a), (1c), (1d) hold, with $\|y\| > \gamma\|u\|$. Applying Lemma 24 to the pair (e_2, y_2), we can produce $\Phi \in \mathscr{L}(\mathcal{V}) \subseteq \mathcal{C}_\Phi$ such that (5) holds, and thus (1b) holds, $(u, y) \in R_{uy}$, and therefore Item (iii) of Theorem 1 fails, as required. All that remains is to prove Lemma 24.

Proof. We begin by considering some special cases.

Special case with $\|e\| = 0$ In this case, we must have $\langle e, y\rangle = 0$ by Cauchy–Schwarz. If $\|y\| = 0$, then define $\Phi = \{(z, x) \mid \|z\| = \|x\| = 0\}$. This is a degenerate case. If $\|y\| > 0$ instead, we have by assumption that

$$M_{22}\|y\|^2 = \left\langle \begin{bmatrix} e \\ y \end{bmatrix}, M \begin{bmatrix} e \\ y \end{bmatrix} \right\rangle \geq 0.$$

Therefore, $M_{22} \geq 0$. Define $\Phi = \{(z, x) \mid \|z\| = 0\}$. Roughly, Φ is the linear relation whose graph is a *vertical line.*

Special case with $\|e\| > 0$ and $\|y\| = 0$ As in the previous case, we must have $\langle e, y\rangle = 0$. By assumption,

$$M_{11}\|e\|^2 = \left\langle \begin{bmatrix} e \\ y \end{bmatrix}, M \begin{bmatrix} e \\ y \end{bmatrix} \right\rangle \geq 0.$$

So, $M_{11} \geq 0$. Define $\Phi x = 0$ and we have

$$\left\langle \begin{bmatrix} x \\ \Phi x \end{bmatrix}, M \begin{bmatrix} x \\ \Phi x \end{bmatrix} \right\rangle = M_{11}\|x\|^2 \geq 0 \quad \text{for all } x \in \mathcal{V}.$$

Henceforth, we will assume that $\|e\| > 0$ and $\|y\| > 0$. Define the normalized vectors $\hat{e} := \frac{e}{\|e\|}$ and $\hat{y} := \frac{y}{\|y\|}$. Also define the normalized inner product $\rho := \langle \hat{e}, \hat{y}\rangle$. Note that by Cauchy–Schwarz, we have $|\rho| \leq 1$.[10]

Special case: $|\rho| = 1$ Define $\Phi x = \rho \frac{\|y\|}{\|e\|} x$ and obtain:

$$\left\langle \begin{bmatrix} x \\ \Phi x \end{bmatrix}, M \begin{bmatrix} x \\ \Phi x \end{bmatrix} \right\rangle = \frac{\|x\|^2}{\|e\|^2} \left\langle \begin{bmatrix} e \\ y \end{bmatrix}, M \begin{bmatrix} e \\ y \end{bmatrix} \right\rangle \geq 0.$$

General case: $|\rho| < 1$ Since M is indefinite, there must exist some $\eta \in \mathbb{F}$ such that $\left[\begin{smallmatrix} 1 \\ \eta \end{smallmatrix}\right]^* M \left[\begin{smallmatrix} 1 \\ \eta \end{smallmatrix}\right] > 0$. For any $x \in \mathcal{V}$, we can write $x = x_{ey} + x_\perp$, where x_{ey} is a linear combination of $\hat{e}$ and $\hat{y}$ and $x_\perp$ is orthogonal to both $\hat{e}$ and $\hat{y}$. This can be computed via Gram–Schmidt:

$$\begin{aligned} x_{ey} &:= \left(\frac{\langle \hat{e}, x\rangle - \rho\langle \hat{y}, x\rangle}{1 - |\rho|^2} \right) \hat{e} + \left(\frac{\langle \hat{y}, x\rangle - \bar{\rho}\langle \hat{e}, x\rangle}{1 - |\rho|^2} \right) \hat{y}, \\ x_\perp &:= x - x_{ey}. \end{aligned}$$

We also have $\|x\|^2 = \|x_{ey}\|^2 + \|x_\perp\|^2$. Define the unit vectors

$$\hat{e}_\perp := \frac{\hat{y} - \rho\hat{e}}{\sqrt{1 - |\rho|^2}} \quad \text{and} \quad \hat{y}_\perp := \frac{\bar{\rho}\hat{y} - \hat{e}}{\sqrt{1 - |\rho|^2}}.$$

[10] Recall that in general, inner products are elements of $\mathbb{F}$, so ρ may be a complex number.

The vectors $\hat{e}_\perp$ and $\hat{y}_\perp$ are orthogonal to $\hat{e}$ and $\hat{y}$, respectively. Write $M_{12} = |M_{12}|e^{i\varphi}$ (polar decomposition). Since $M_{21} = \overline{M}_{12}$, we have the identity: $e^{-2i\varphi}M_{12} = M_{21}$.

Finally, define Φ as:

$$\Phi x := \frac{\|y\|}{\|e\|}\Big(\langle \hat{e}, x_{ey}\rangle \hat{y} + e^{-2i\varphi}\langle \hat{e}_\perp, x_{ey}\rangle \hat{y}_\perp\Big) + \eta\, x_\perp.$$

The function Φ is linear. The bracketed term lies in the span of $\hat{e}$ and $\hat{y}$ and performs an isometry that maps $\hat{e} \mapsto \hat{y}$, followed by the scaling $\frac{\|y\|}{\|e\|}$. This ensures that $\Phi e = y$. The remainder of Φx acts on the part of x orthogonal to the span of $\hat{e}$ and $\hat{y}$, and simply scales by η. One can readily check that $\|\Phi x_{ey}\| = \frac{\|y\|}{\|e\|}\|x_{ey}\|$. and $\operatorname{Re}\Big(M_{12}\langle x_{ey}, \Phi x_{ey}\rangle\Big) = \frac{\|y\|}{\|e\|}\|x_{ey}\|^2 \operatorname{Re}(M_{12}\rho)$. Thus, $\begin{bmatrix} x \\ \Phi x\end{bmatrix} = \begin{bmatrix} x_{ey} \\ \Phi x_{ey}\end{bmatrix} + \begin{bmatrix} x_\perp \\ \eta x_\perp\end{bmatrix}$ and

$$\left\langle \begin{bmatrix} x \\ \Phi x\end{bmatrix}, M\begin{bmatrix} x \\ \Phi x\end{bmatrix}\right\rangle = \left\langle \begin{bmatrix} x_{ey} \\ \Phi x_{ey}\end{bmatrix}, M\begin{bmatrix} x_{ey} \\ \Phi x_{ey}\end{bmatrix}\right\rangle + \left\langle \begin{bmatrix} x_\perp \\ \eta x_\perp\end{bmatrix}, M\begin{bmatrix} x_\perp \\ \eta x_\perp\end{bmatrix}\right\rangle.$$

The first term simplifies to

$$\begin{aligned}
&\left\langle \begin{bmatrix} x_{ey} \\ \Phi x_{ey}\end{bmatrix}, M\begin{bmatrix} x_{ey} \\ \Phi x_{ey}\end{bmatrix}\right\rangle \\
&= M_{12}\|x_{ey}\|^2 + 2\operatorname{Re}\Big(M_{12}\langle x_{ey}, \Phi x_{ey}\rangle\Big) + M_{22}\|\Phi x_{ey}\|^2 \\
&= \|x_{ey}\|^2\left(M_{11} + 2\operatorname{Re}(M_{12}\rho)\frac{\|y\|}{\|e\|} + M_{22}\frac{\|y\|^2}{\|e\|^2}\right) \\
&= \frac{\|x_{ey}\|^2}{\|e\|^2}\left(M_{11}\|e\|^2 + 2\operatorname{Re}\left(M_{12}\langle e, y\rangle\right) + M_{22}\|y\|^2\right) \\
&= \frac{\|x_{ey}\|^2}{\|e\|^2}\left\langle \begin{bmatrix} e \\ y\end{bmatrix}, M\begin{bmatrix} e \\ y\end{bmatrix}\right\rangle \ge 0.
\end{aligned}$$

The second term simplifies to

$$\left\langle \begin{bmatrix} x_\perp \\ \eta x_\perp\end{bmatrix}, M\begin{bmatrix} x_\perp \\ \eta x_\perp\end{bmatrix}\right\rangle = \|x_\perp\|^2\begin{bmatrix} 1 \\ \eta\end{bmatrix}^* M\begin{bmatrix} 1 \\ \eta\end{bmatrix} \ge 0.$$

Therefore, we have $\left\langle \begin{bmatrix} x \\ \Phi x\end{bmatrix}, M\begin{bmatrix} x \\ \Phi x\end{bmatrix}\right\rangle \ge 0$, as required. ∎

B Proofs for extended spaces

First, we state a useful result that connects norms in $\mathcal{L}_2$ with truncated norms in $\mathcal{L}_{2\mathrm{e}}$.

Proposition 25. *Suppose $H : \mathcal{L}_{2\mathrm{e}} \to \mathcal{L}_{2\mathrm{e}}$ is causal, and for all $x \in \mathcal{L}_2$, we have $\|Hx\| \le \gamma\|x\|$. Then for all $x \in \mathcal{L}_{2\mathrm{e}}$ and $T \ge 0$, we have $\|Hx\|_T \le \gamma\|x\|_T$.*

Proof. This result appears for example in [31, Lem. 6.2.11]. The proof is short so we reproduce it here. Let $x \in \mathcal{L}_{2\mathrm{e}}$, and then: $\|Hx\|_T = \|Hx_T\|_T \le \|Hx_T\| \le \gamma\|x_T\| = \gamma\|x\|_T$. ∎

B.1 Proof of Corollary 15

Choose $\mathcal{V} = \mathcal{L}_{2\mathrm{e}}$ with inner product $\langle \cdot, \cdot \rangle_T$ and let $\mathcal{C}_G \subseteq \mathscr{F}(\mathcal{L}_{2\mathrm{e}})$ and $\mathcal{C}_\Phi \subseteq \mathscr{F}(\mathcal{L}_{2\mathrm{e}})$.

To prove (i) $\Longrightarrow$ (ii), note that from the proof of Theorem 1, the gain γ only depends on the choice of M and N, and not on the choice of semi-inner product (choice of T). Likewise, fixing Φ and $u \in \mathcal{L}_2$ yields the same R_{uy} for all T. Therefore, we have $\|y\|_T \le \gamma \|u\|_T$ for all $T \ge 0$ and γ, y are independent of T. Since $u \in \mathcal{L}_2$, letting $T \to \infty$ implies $y \in \mathcal{L}_2$ and we obtain (11).

To prove (i) $\Longleftarrow$ (ii), suppose that for all (u, y, e) with $u \in \mathcal{L}_2$ satisfying (9), (1a), (1c), (1d), we have $\|y\| \le \gamma \|u\|$. Since $\mathcal{C}_G$ and $\mathcal{C}_\Phi$ are subsets of $\mathscr{F}(\mathcal{L}_{2\mathrm{e}})$, then G and Φ are causal and so the closed-loop map R_{uy} is causal whenever the interconnection is well-posed [29, Prop. 1.2.14]. By Proposition 25, we have $\|y_T\| \le \gamma \|u\|_T$. Apply Theorem 1 as before to obtain (8). From the way N is constructed in the proof of Theorem 1 (Step 5 in Appendix A.1), N is independent of T and therefore the same N may be used for all T.

Proving (ii) $\Longrightarrow$ (iii) is similar to the corresponding proof in Theorem 1. Consider some $\Phi \in \mathcal{C}_\Phi \subseteq \mathscr{F}(\mathcal{L}_{2\mathrm{e}})$ for which (10) holds. Next, let (u, y, e) be the solution of (1). In particular, (1b) holds, so setting $\xi = e_2$ in (10), we obtain (9) and the rest of the proof follows as in Section 2.1. ■

B.2 Proof of Lemma 17

Define $G_s := (P_{11}G + P_{12})(P_{21}G + P_{22})^{-1}$. Suppose Item 1 holds. Substituting the factorization for N into (12), we obtain

$$\left\langle \begin{bmatrix} P_{11}G\xi + P_{12}\xi \\ P_{21}G\xi + P_{22}\xi \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_{11}G\xi + P_{12}\xi \\ P_{21}G\xi + P_{22}\xi \end{bmatrix} \right\rangle_T \ge 0$$
$$\Longleftrightarrow \; \|P_{11}G\xi + P_{12}\xi\|_T \le \|P_{21}G\xi + P_{22}\xi\|_T.$$

Let $\zeta = P_{21}G\xi + P_{22}\xi$. Eliminating ξ from the above, we obtain: $\|G_s\zeta\|_T \le \|\zeta\|_T$ Taking the limit $T \to \infty$, we conclude that G_s is bounded in the $\mathcal{L}_2$ norm.

Whenever $P_{21}G\xi + P_{22}\xi \in \mathcal{L}_2$, boundedness of G_s implies that $P_{11}G\xi + P_{12}\xi \in \mathcal{L}_2$. In such a case, we can take the limit $T \to \infty$ in (12) and $\langle \cdot, \cdot \rangle_T$ becomes $\langle \cdot, \cdot \rangle$, the inner product on $\mathcal{L}_2$. Applying Parseval's theorem yields

$$\left\langle \begin{bmatrix} P_{11}\hat{G} + P_{12} \\ P_{21}\hat{G} + P_{22} \end{bmatrix} \hat{\xi}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_{11}\hat{G} + P_{12} \\ P_{21}\hat{G} + P_{22} \end{bmatrix} \hat{\xi} \right\rangle \ge 0.$$

Due to our freedom in choosing $\hat{\xi}$, we conclude that

$$\begin{bmatrix} P_{11}\hat{G}(\omega) + P_{12} \\ P_{21}\hat{G}(\omega) + P_{22} \end{bmatrix}^* \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_{11}\hat{G}(\omega) + P_{12} \\ P_{21}\hat{G}(\omega) + P_{22} \end{bmatrix} \ge 0$$

holds for almost all $\omega \in \mathbb{R}$. Substituting the factorization for N, we obtain Item 2. Conversely, we can begin from (13) and apply the steps in reverse together with Proposition 25 and we recover Item 1. ■

B.3 Proof of Theorem 18

We proceed by contradiction. Suppose Item (ii) in Corollary 15 fails. Since (i) $\Longleftrightarrow$ (ii), this is equivalent to (i) failing. By Lemma 17, either $G_s := (P_{11}G + P_{12})(P_{21}G + P_{22})^{-1}$ does not map

$\mathcal{L}_2 \to \mathcal{L}_2$, or there must exist $\omega_0 \in \mathbb{R}$ such that

$$\begin{bmatrix} \hat{G}(\omega_0) \\ 1 \end{bmatrix}^* N \begin{bmatrix} \hat{G}(\omega_0) \\ 1 \end{bmatrix} < 0. \tag{33}$$

If the former is true, choose the following static gain Φ and (frequency domain) input signals. If $P_{21} \neq 0$ and $P_{22} \neq 0$, let

$$\hat{u}_1 = -\tfrac{P_{11}}{P_{21}}\hat{\eta}, \qquad \hat{u}_2 = -\tfrac{P_{12}}{P_{21}}\hat{\eta}, \qquad \Phi = -\tfrac{P_{21}}{P_{22}}. \tag{34}$$

This is a valid choice of Φ because it satisfies (10) upon substituting the factorization for M. Substituting Φ, u_1, u_2 into the loop equations (1), we obtain $\hat{y}_2 = \hat{G}_s\hat{\eta}$. So the closed-loop map is not stable. If $P_{21} = 0$, we have $P = I$, so $G_s = G$. Then simply pick $\Phi = 0$ and the closed-loop map is unstable. If $P_{22} = 0$, use the choice (34) with $P_{22} = \varepsilon > 0$. Since $\hat{y}_2 \to \hat{G}_s\hat{\eta}$ as $\varepsilon \to 0$, then For ε sufficiently small, the map from $\hat{\eta}$ to $\hat{y}_2$ is unstable.

If (33) holds instead, then fix $\gamma > 0$ and follow the construction used in Lemma 7 to prove $\neg$(i) $\implies \neg$(ii), except use the complex numbers $\xi \mapsto 1$ and $G\xi \mapsto \hat{G}(\omega_0)$ to seed the construction in (25). This results in $u_1, u_2, y_1, y_2, e_1, e_2 \in \mathbb{C}$ that satisfy (1a), (1c), (1d), $\|y\| > \gamma\|u\|$, and

$$\begin{bmatrix} e_2 \\ y_2 \end{bmatrix}^* M \begin{bmatrix} e_2 \\ y_2 \end{bmatrix} \geq 0. \tag{35}$$

Now view the constructed (y, u, e) as *phasors*. For example, y_1 is multiplied by $e^{j\omega_0 t}$ and represents a sinusoidal input with frequency ω_0 and magnitude and phase equal to those of y_1. The other signals in (y, u, e) are also multiplied by $e^{j\omega_0 t}$ and interpreted similarly.

The set of ω_0 that satisfy (33) is open, so we may pick ω_0 such that $e_2 \neq 0$. Choose $\Phi = \frac{y_2}{e_2} = re^{-j\theta}$ where $r > 0$ and $\theta \in [0, 2\pi)$ are the polar representation. So Φ is a static gain r cascaded with a pure delay of $\frac{\theta}{\omega_0}$. By construction, the signals (y, u, e) satisfy (1) and so $y = R_{uy}u$. Now there are two possible cases. If the closed-loop map R_{uy} is unstable, then we have shown that (iii) fails, as required, for an unstable system cannot have a finite $\mathcal{L}_2$ gain. If the closed-loop map R_{uy} is instead stable, then since (35) holds for the phasors (u, y, e) and multiplying each phasor by $e^{j\omega_0 t}$ does not change the instantaneous value of (35), we have

$$\left\langle \begin{bmatrix} e_2 \\ y_2 \end{bmatrix}, M \begin{bmatrix} e_2 \\ y_2 \end{bmatrix} \right\rangle_T \geq 0 \quad \text{for all } T \geq 0$$

and $\|y\|_T > \gamma\|u\|_T$ for the time-domain sinusoids (u, y, e). Due to stability of R_{uy}, this sinusoidal fixed point of the dynamics is stable, and has gain $\gamma > 0$, which was arbitrarily chosen. It follows that (iii) fails, as required. ∎

Alternative proof. Consider the set $\mathcal{V} = \mathcal{L}_2(\mathbb{R})$ of frequency domain signals (either continuous or discrete time). Let

$$\mathcal{C}_G \subseteq \left\{ \mathcal{M}_{\hat{G}} \;\middle|\; \hat{G} \in \mathcal{L}_\infty \right\} \subseteq \mathcal{C}_\Phi,$$

where $\mathcal{M}_{\hat{G}}$ is the multiplication operator corresponding to the essentially bounded frequency response $\hat{G}$. Applying Theorem 1, Equation (3) becomes

$$\left\langle \begin{bmatrix} \hat{G}(\omega)\hat{\xi}(\omega) \\ \hat{\xi}(\omega) \end{bmatrix}, N \begin{bmatrix} \hat{G}(\omega)\hat{\xi}(\omega) \\ \hat{\xi}(\omega) \end{bmatrix} \right\rangle \geq 0 \quad \text{for all } \hat{\xi} \in \mathcal{L}_2(\mathbb{R}),$$

which is equivalent to

$$\begin{bmatrix} \hat{G}(\omega) \\ 1 \end{bmatrix}^* N \begin{bmatrix} \hat{G}(\omega) \\ 1 \end{bmatrix} \geq 0 \quad \text{for almost all } \omega \in \mathbb{R},$$

which is equivalent to (8) via Lemma 17. Applying Theorem 12, we conclude that a linear function Φ may be used to certify the necessary direction of Corollary 15 and the rest of the proof proceeds as above. ∎

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