

Information Structures of the Kalman Filter for the Elastic Wave Equation

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Abstract: We study the Kalman Filter for the linear elastic wave equation over the real line with spatially distributed partial state measurements. The dynamics of the filter are described by a spatial convolution operator with asymptotic exponential spatial decay rate. This decay rate dictates how measurements from different spatial locations must be exchanged to implement the filter: faster spatial decay implies local measurements are more relevant and the filter is more “decentralized”; slower decay implies farther measurements also become relevant and the filter is more “centralized”. Using dimensional analysis, we demonstrate that this decay rate is a function of one dimensionless group defined from system parameters, such as wave speed and noise variances. We find a critical value of such dimensionless group for which the Kalman Filter is completely decentralized.

Keywords: Distributed estimation; Infinite-dimensional systems; Elastic waves.

1. INTRODUCTION

Kalman filtering of infinite dimensional systems has been an active area of research for decades (see e.g., Curtain, 1975, for a review). More recent efforts have addressed the question of sensor location selection for filtering error variance minimization in distributed parameter systems. For example, Zhang and Morris (2018) explored optimal sensor placement and the trade-off between the number of sensors and their quality, and Demetriou and Uciński (2011) proposed a simple real time guidance scheme for mobile sensors used to enhance state estimation of a spatially distributed process described by a linear Partial Differential Equation (PDE). A related problem is that of designing measurement methods for system identification of PDEs (see Uciński, 2005, for a monograph on the topic).

Our problem set-up is different. We are interested in the state estimation problem for linear elastic wave dynamics in which only displacement measurements are available (i.e., with partial state information), but this measurement is accessible over the whole spatial domain. We analyze the information structures of Kalman filters in this setting and further consider how plant’s parameters determine the degree of spatial localization of the filter. The practical relevance of this question includes e.g., to leverage modular embedded sensing, computation, and communication for locomotion of soft robots through elastic intelligent

materials (see e.g., Correll et al., 2014; Kim et al., 2013), and to better understand the role of proprioception in the coordination for crawling locomotion of soft-bodied organisms, which can be achieved by sustaining peristaltic waves along their bodies (e.g., Pehlevan et al., 2016; Saga and Nakamura, 2004; Tanaka et al., 2012).

Although the mentioned applications have finite spatial extent, in this work we examine the wave equation over an *unbounded* spatial domain. This assumption allows us to derive analytic bounds on the asymptotic spatial decay rate of the Kalman filter in terms of system parameters, which in turn determines the information structures of the filter. This may provide insight to observed behaviors in finite spatial domain settings as the unbounded setting is a useful idealization of the large but finite setting in certain cases (see e.g., Tegling and Sandberg, 2017; Curtain et al., 2010). Indeed, for certain problems it has been shown that the solution to the finite extent problem with boundary conditions is that of the spatially-invariant counterpart plus a “correction term” at the edges (see Epperlein and Bamieh, 2016).

In this work, we follow the Kalman filtering framework introduced in (Balakrishnan, 1981, Ch. 6). Related works include Lee et al. (2011), where the effect of modeling error in the quality of the state prediction of the Kalman filter for a linear wave equation is analyzed, and the

seminal work of Bamieh et al. (2002) on optimal control of spatially-invariant systems over L^2 spaces. Under mild assumptions, Bamieh et al. (2002) showed that solutions to operator Riccati equations with spatially-invariant coefficients are spatially-invariant themselves. Hence, optimal state estimator dynamics for spatially-invariant plants are also spatially-invariant and described by a spatial convolution operator. The kernel of such convolution decays exponentially in space. The work of Bamieh et al. (2002) on spatially-invariant systems was extended beyond the L^2 setting to the case of a Sobolev state space for LQR control problems in Epperlein and Bamieh (2014) and Jensen et al. (2020). Generalizations to the spatially-varying setting include Motee and Jadbabaie (2008), which demonstrated the spatial decay of optimal feedback operators for the class of spatially decaying plants. The Kalman filtering problem for spatially-invariant plants has also attracted some attention: Henningsson and Rantzer (2007) studied scalable distributed implementations of the Kalman filter for a finite-dimensional circulant mass-spring system, and Arbelaz et al. (2020) characterized how the statistical properties of the noise define the spatial localization of the Kalman filter for a spatially-invariant diffusion process over the real line.

In this work, we build upon insights from Jensen et al. (2020) and Arbelaz et al. (2020) to characterize how plant's parameters define the information structures of the Kalman filter for an elastic wave equation. Through dimensional analysis, we determine a single dimensionless parameter that completely characterizes the universal spatial decay rate of the Kalman Gain in our setting and identify a critical value of this parameter for which the filter becomes completely decentralized.

The paper is organized as follows. Section 2 introduces notation and mathematical background necessary to follow our exposition. The plant and formulation of the Kalman filtering problem are presented in Section 3, after a brief prelude to dimensional analysis. The main result on the spatial localization of the Kalman filter is provided in Section 4. Our results and conclusions are discussed in Section 5.

2. NOTATION & MATHEMATICAL PRELIMINARIES

Let H and H' denote two Hilbert spaces. We denote the space of linear operators from H to H' by $\mathcal{L}(H, H')$ and write $\mathcal{L}(H, H) = \mathcal{L}(H)$. The *domain* of $A \in \mathcal{L}(H, H')$ is denoted by $\mathcal{D}(A) \subset H$ and the *adjoint* of A is the operator $A^\dagger \in \mathcal{L}(H', H)$ that satisfies

$$\langle Af, g \rangle_{H'} = \langle f, A^\dagger g \rangle_H \quad (1)$$

for all $f \in \mathcal{D}(A)$ and all $g \in \mathcal{D}(A^\dagger)$. A is *self-adjoint* if $\mathcal{D}(A) = \mathcal{D}(A^\dagger)$ and $A = A^\dagger$. This somewhat non-standard notation is used to distinguish adjoints from the matrix complex conjugate transpose, which we denote by $(\cdot)^*$.

$A \in \mathcal{L}(H, H')$ is *bounded* if $\|A\| := \sup_{\|f\|_H=1} \|Af\|_{H'}$ is finite.

Spatio-temporal Signals: We consider a spatially distributed system over the unbounded spatial domain \mathbb{R} , whose dynamics are described using (possibly vector-valued) *spatio-temporal signals* $\psi = \psi(x, t)$, with $x \in \mathbb{R}$ the

spatial variable and $t \in [0, \infty)$ the temporal variable. For each t , $\psi(\cdot, t)$ is an element of a Hilbert space. Denoting $\psi(t) := \psi(\cdot, t)$, ψ can be thought of as a Hilbert space valued temporal signal. For the problem of interest, such temporal signals take values in one of the following two Hilbert spaces (or Hilbert direct sums of these spaces):

- $L_n^2(\mathbb{R})$ denotes the set of square-integrable functions from \mathbb{R} to \mathbb{C}^n equipped with the inner product

$$\langle f, g \rangle_{L_n^2(\mathbb{R})} := \int_{x \in \mathbb{R}} g(x)^* f(x) dx. \quad (2)$$

- For $\alpha > 0$, $\mathcal{H}_\alpha(\mathbb{R})$ denotes the Sobolev space of weakly differentiable functions from \mathbb{R} to \mathbb{C} equipped with the inner product

$$\langle f, g \rangle_{\mathcal{H}_\alpha(\mathbb{R})} := \langle f, g \rangle_{L^2(\mathbb{R})} + \alpha^2 \langle \partial_x f, \partial_x g \rangle_{L^2(\mathbb{R})} \quad (3)$$

To simplify notation, we often write $L^2 = L_n^2(\mathbb{R})$ and $\mathcal{H}_\alpha = \mathcal{H}_\alpha(\mathbb{R})$. We denote the spatial Fourier transform of spatio-temporal signals using hats:

$$\hat{\psi}_k(t) = (\mathcal{F}\psi)(k, t) := \frac{1}{\sqrt{2\pi}} \int_{x \in \mathbb{R}} \psi(x, t) e^{-ixk} dx, \quad (4)$$

where $k \in \mathbb{R}$ is the spatial frequency. For vector-valued signals, the transform is defined component-wise. It is well known (Plancherel Theorem) that the Fourier transform is an isometric isomorphism from L^2 to itself, i.e.,

$$\langle f, g \rangle_{L^2(\mathbb{R})} = \langle \hat{f}, \hat{g} \rangle_{L^2(\mathbb{R})}. \quad (5)$$

More generally, the Fourier transform is an isometric isomorphism from Sobolev spaces, such as \mathcal{H}_α to *weighted L^2 spaces*.

Definition 1. For $W : \mathbb{R} \rightarrow \mathbb{C}^{n \times n}$ with $W(k)$ nonsingular for each k , define the weighted L^2 space, L_W , by the inner product

$$\langle \hat{f}, \hat{g} \rangle_{L_W} := \int_{k \in \mathbb{R}} \hat{f}(k)^* W(k) \hat{g}(k) dk. \quad (6)$$

W is the *spatial frequency weighting function* of L_W .

The following result follows from e.g. Epperlein and Bamieh (2014).

Proposition 2. The Fourier transform is an isometric isomorphism from the Sobolev space $\mathcal{H}_\alpha(\mathbb{R})$ to $\hat{\mathcal{H}}_\alpha(\mathbb{R}) = L_w$, the weighted L^2 space defined by the spatial frequency weighting function

$$w(\alpha; k) := 1 + \alpha^2 k^2, \quad (7)$$

i.e. $\langle f, g \rangle_{\mathcal{H}_\alpha(\mathbb{R})} = \langle \hat{f}, \hat{g} \rangle_{\hat{\mathcal{H}}_\alpha(\mathbb{R})}$.

It follows from Proposition 2 that if $\psi(t) = \psi(\cdot, t)$ is an L^2 (resp. \mathcal{H}_α) valued signal, then $\hat{\psi}(t) = \hat{\psi}(\cdot, t)$ is an L^2 (resp. $\hat{\mathcal{H}}_\alpha$) valued signal. It is straightforward to extend to signals that are in Hilbert space direct sums of such spaces.

Spatially-Invariant Operators: Let H and H' denote two Hilbert spaces given by L^2 , \mathcal{H}_α , or direct sums of these two spaces.

To each $z \in \mathbb{R}$, define the associated translation by z operator by $(T_z f)(x) = f(x-z)$. $A \in \mathcal{L}(H, H')$ is *spatially-invariant* if it commutes with all translation operators, i.e.

$$T_z|_{\mathcal{D}(A)} A = A T_z|_{\mathcal{D}(A)}, \text{ for all } z \in \mathbb{R},$$

with $T_z|_{\mathcal{D}(A)}$ denoting the restriction of the translation operator T_z to the domain of A .

Example 3. An operator A of the form

$$(Af)(x) = \int_{\xi \in \mathbb{R}} a(\xi) f(x - \xi) d\xi, \quad (8)$$

is spatially-invariant. Here a is the *convolution kernel* of A . Allowing a to include generalized functions, e.g. dirac deltas and their derivatives, differential operators may be written in form (8).

Definition 4. The convolution kernel a *decays exponentially* if $a(x)e^{\beta|x|} \rightarrow 0$ as $|x| \rightarrow \infty$ for some $\beta > 0$. We call the largest of such β the *decay rate* of the corresponding operator A .

A spatially-invariant operator A is “diagonalized” by the spatial Fourier transform in the sense that $\mathcal{F}A\mathcal{F}^{-1}$ is a *multiplication operator*, that is, $(\mathcal{F}A\mathcal{F}^{-1}\hat{f})(k) = \hat{A}(k)\hat{f}(k)$. We refer to \hat{A} as the *symbol* of the operator $\mathcal{F}A\mathcal{F}^{-1}$. When A is of the form (8), this symbol \hat{A} is the Fourier transform of the convolution kernel a of A :

$$\hat{A}_k = \hat{A}(k) = \frac{1}{\sqrt{2\pi}} \int_{x \in \mathbb{R}} a(x) e^{-ixk} dx. \quad (9)$$

With some abuse of notation we often use \hat{A} to denote the operator $\mathcal{F}A\mathcal{F}^{-1}$ as well. Moreover, under certain conditions, the decay rate of A can be determined through the region of analyticity of its symbol \hat{A} according to the following theorem.

Theorem 5. (adapted from Hörmander (2015)). An *extension* of the symbol of a multiplication operator \hat{A} to the complex plane, denoted by \hat{A}^{ext} , is constructed by replacing each k in $\hat{A}(k)$ by $(-iz)$ with $z \in \mathbb{C}$, to give $\hat{A}^{\text{ext}}(z)$. Let $\eta > 0$. If \hat{A}^{ext} is analytic and satisfies a polynomial growth bound on the strip

$$\Gamma_\eta = \{z \in \mathbb{C}; |\Re(z)| < \eta\} = (-\eta, \eta) + i\mathbb{R} \subset \mathbb{C},$$

then $A = \mathcal{F}^{-1}\hat{A}\mathcal{F}$ has decay rate β for any $\beta < \eta$.

A spatially-invariant operator A is self-adjoint if and only if $\hat{A} = (\mathcal{F}A\mathcal{F}^{-1})$ is, and the operator norm satisfies $\|A\| = \|\hat{A}\|$. Symbols of adjoints of interest are stated in the following proposition, which follows from Epperlein and Bamieh (2014).

Proposition 6. Let $\hat{A} \in \mathcal{L}(L_V, L_W)$ be a multiplication operator between two weighted L^2 spaces. Then the adjoint of \hat{A} is also a multiplication operator, with symbol given by $\hat{A}^\dagger(k) = V(k)^{-1}\hat{A}(k)^*W(k)$.

3. PROBLEM FORMULATION

This work is concerned with the Kalman filtering problem for the elastic wave equation in an unbounded spatial domain with partial and noisy state measurement:

$$\frac{\partial^2 p}{\partial t^2}(x, t) = c^2 \frac{\partial^2 p}{\partial x^2}(x, t) + d(x, t), \quad (10a)$$

$$y(x, t) = p(x, t) + n(x, t), \quad (10b)$$

with $t > 0$, $x \in \mathbb{R}$, and $c > 0$ denotes the wave speed. The displacement p , process disturbance d , noisy measurement y , and measurement noise n are each spatio-temporal signals. We write (10) in state space form¹ as

¹ Note that this choice of state space representation is non-unique.

$$\frac{d}{dt} \begin{bmatrix} \mathbf{p}(t) \\ \partial_t \mathbf{p}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ c^2 \partial_x^2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}(t) \\ \partial_t \mathbf{p}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \mathbf{d}(t), \quad (11)$$

$$\mathbf{y}(t) = [I \ 0] \begin{bmatrix} \mathbf{p}(t) \\ \partial_t \mathbf{p}(t) \end{bmatrix} + \mathbf{n}(t).$$

We focus on the case that \mathbf{d} and \mathbf{n} are white noise mutually independent processes, as formalized in (Balakrishnan, 1981, Ch. 6), with covariance operators

$$R_d = \sigma_d^2 \cdot I \quad \text{and} \quad R_n = \sigma_n^2 \cdot I, \quad (12)$$

respectively.

The operator $\begin{bmatrix} 0 & I \\ c^2 \partial_x^2 & 0 \end{bmatrix}$ generates a C_0 -semigroup of operators on the Hilbert space $\mathcal{X}_\alpha := \mathcal{H}_\alpha(\mathbb{R}) \oplus \frac{\alpha}{c} L^2(\mathbb{R})$, for any choice of $\alpha > 0$, and its domain

$$\mathcal{D} \left(\begin{bmatrix} 0 & I \\ c^2 \partial_x^2 & 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathcal{X}_\alpha; f_1', f_2'' \in L^2(\mathbb{R}) \right\}$$

is dense in \mathcal{X}_α . The corresponding measure of the state is

$$\left\| \begin{bmatrix} \mathbf{p}(t) \\ \partial_t \mathbf{p}(t) \end{bmatrix} \right\|_{\mathcal{X}_\alpha}^2 = \|\mathbf{p}(t)\|_{L^2}^2 + \alpha^2 \|\partial_x^2 \mathbf{p}(t)\|_{L^2}^2 + \frac{\alpha^2}{c^2} \|\partial_t \mathbf{p}\|_{L^2}^2.$$

The parameter α determines the relative penalty between the first state component and its spatial derivative.

3.1 Dimensional Analysis

Before the formulation of our main result, we non-dimensionalize the plant. This will allow us to group the different parameters of the problem in a smaller number of dimensionless parameters, easing sensitivity analysis and interpretation of the results. Since α sets a lengthscale and c sets a speed, we define the dimensionless spatial (χ) and temporal (τ) variables:

$$\chi := \frac{1}{\alpha} x \quad \text{and} \quad \tau := \frac{c}{\alpha} t. \quad (13)$$

Noting also that σ_n has the same dimension as the state and σ_d has dimensions of state per squared time, we define the dimensionless disturbance and noise signals

$$\phi(\tau) := \frac{c^2}{\alpha^2 \sigma_d} \mathbf{p}(\tau) \quad \text{and} \quad \gamma(\tau) := \frac{1}{\sigma_n} \mathbf{y}(\tau). \quad (14)$$

Then the non-dimensional counterpart of the dynamics (11) are

$$\frac{d}{d\tau} \begin{bmatrix} \phi(\tau) \\ \partial_\tau \phi(\tau) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ \partial_\chi^2 & 0 \end{bmatrix}}_{=:A} \begin{bmatrix} \phi(\tau) \\ \partial_\tau \phi(\tau) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{=:B} \boldsymbol{\rho}(\tau), \quad (15a)$$

$$\gamma(\tau) = \underbrace{\begin{bmatrix} 1 \\ \pi_e \end{bmatrix}}_{=:C} \begin{bmatrix} \phi(\tau) \\ \partial_\tau \phi(\tau) \end{bmatrix} + \boldsymbol{\eta}(\tau), \quad (15b)$$

where π_e is the dimensionless parameter

$$\pi_e := \frac{c^2 \sigma_n}{\alpha^2 \sigma_d} > 0. \quad (16)$$

$\boldsymbol{\rho} := \frac{1}{\sigma_d} \mathbf{d}$ and $\boldsymbol{\eta} := \frac{1}{\sigma_n} \mathbf{n}$ are mutually independent white noise processes with identity covariance operators.

The Sobolev norm on the transformed state is non-dimensionalized accordingly as

$$\left\| \begin{bmatrix} \phi(\tau) \\ \partial_\tau \phi(\tau) \end{bmatrix} \right\|^2 = \|\phi(\tau)\|_{L^2}^2 + \|\partial_\chi \phi(\tau)\|_{L^2}^2 + \|\partial_\tau \phi(\tau)\|_{L^2}^2,$$

i.e. the nondimensional state $\psi(\tau) = [\phi(\tau) \partial_\tau \phi(\tau)]^T \in \mathcal{X}' := \mathcal{H}_1 \oplus L^2$. A generates a C_0 -semigroup on \mathcal{X}' , and $B \in \mathcal{L}(L^2, \mathcal{X}')$ and $C \in \mathcal{L}(\mathcal{X}', \mathcal{H}_1)$ are bounded operators.

From (15b) we see that π_e admits physical interpretation as the *noise-to-signal ratio* in the non-dimensional observations: the smaller π_e , the higher the signal content in the measurement compared to the noise.

3.2 Kalman Filtering

The Kalman Filtering problem of interest is to characterize the steady-state dynamics of the state estimate

$$\tilde{\psi}(\tau) := \mathbb{E} [\psi(\tau) \mid \{\gamma(s); s \leq \tau\}]. \quad (17)$$

It is straightforward to confirm² exponential stabilizability of (A, B) and exponential detectability of (A, C) . We may then derive the state estimator dynamics by following the framework of (Balakrishnan, 1981, Sec. 6.8). Defining the (nondimensional) estimation error

$$e(t) := \psi(t) - \tilde{\psi}(t), \quad (18)$$

the dynamics of the estimate (17) are given by

$$\frac{d}{d\tau} \tilde{\psi}(\tau) = (A - LC) \tilde{\psi}(\tau) + L\gamma(\tau) \quad (19)$$

where the operator

$$L := PC^\dagger \quad (20)$$

is the *Kalman Filter Gain* and the operator

$$P = \lim_{\tau \rightarrow \infty} \mathbb{E}[e(\tau)e(\tau)^*] \quad (21)$$

is the self-adjoint, positive definite solution to the infinite dimensional algebraic Riccati equation (ARE)

$$0 = \langle Ph, A^\dagger \ell \rangle + \langle A^\dagger h, P\ell \rangle + \langle BB^\dagger h, \ell \rangle - \langle PC^\dagger CPh, \ell \rangle, \quad (22)$$

where $\ell, h \in \mathcal{D}(A^\dagger)$ and $\langle \cdot, \cdot \rangle$ is the inner product on \mathcal{X}' .

Proposition 7. The Kalman Filter Gain L is a spatially-invariant operator. In the nondimensional spatial frequency domain it is represented by a multiplication operator with symbol

$$\hat{L}_\kappa = \begin{bmatrix} \sqrt{2\pi_e \hat{L}_0(\kappa)} \\ \hat{L}_0(\kappa) \end{bmatrix}, \quad (23)$$

where

$$\hat{L}_0(\kappa) = -\pi_e \kappa^2 + \sqrt{\pi_e^2 \kappa^4 + \kappa^2 + 1}, \quad (24)$$

and $\kappa := \alpha\kappa$ is the nondimensional spatial frequency.

Proof. See Appendix.

Remark 8. The dimensional Kalman Gain, which corresponds to the dynamics of the estimate of the original dimensional state $[\mathbf{p} \partial_t \mathbf{p}]^T$, can be recovered from L as:

$$\hat{\Lambda}_\kappa = \frac{c}{\alpha\pi_e} \begin{bmatrix} 1 & 0 \\ 0 & (c/\alpha) \end{bmatrix} \hat{L}_\kappa. \quad (25)$$

4. SPATIAL LOCALIZATION OF THE KALMAN GAIN FOR THE ELASTIC WAVE EQUATION

The Fourier transformed dynamics of the Kalman Filter (19) are decoupled in κ :

² A formal definition of exponential stabilizability and detectability can be found in e.g. Curtain and Zwart (2020) and a method for checking stability, stabilizability and detectability in the spatially-invariant setting is provided in Epperlein (2014).

$$\frac{d\hat{\psi}_\kappa}{d\tau}(\tau) = (\hat{A}_\kappa - \hat{L}_\kappa \hat{C}_\kappa) \hat{\psi}_\kappa(\tau) + \hat{L}_\kappa \hat{\gamma}_\kappa(\tau). \quad (26)$$

By the convolution theorem, the pointwise multiplication $\hat{L}_\kappa \hat{\gamma}_\kappa$ transforms to a spatial convolution in the physical domain. Hence, the spatial decay rate of the kernel L dictates how measurements must be exchanged within the system to implement the filter: if the spatial decay of L is fast, at each spatial location measurements from its neighborhood are more relevant for the filter than those further away; if slow, measurements from further away become relevant as well. Under mild assumptions, Bamieh et al. (2002) proved that L decays at least exponentially in space, which implies that the Kalman Filter has an inherent degree of spatial localization and is suitable for a distributed implementation.

In this section we study the effect of system's parameters on the spatial decay rate of the Kalman Filter Gain (20), that is, how the interplay between the parameters determines the level of spatial localization of the filter.

Theorem 9. Consider the Kalman Filter (19) for the linear elastic wave equation and observations (15) perturbed by white noise. The universal spatial decay rate β of the Kalman Filter Gain in this setting is

$$\beta = \begin{cases} \frac{1}{\sqrt{2\pi_e}} \sqrt{1 - \sqrt{1 - 4\pi_e^2}} & \text{if } 0 < \pi_e < \frac{1}{2} \\ \frac{1}{\sqrt{2\pi_e}} \sqrt{\pi_e + \frac{1}{2}} & \text{if } \pi_e > \frac{1}{2} \end{cases}, \quad (27)$$

where π_e is the dimensionless parameter defined in (16). At the critical value $\pi_e^* = \frac{1}{2}$ the Kalman filter is *completely decentralized*.

Proof. We derive the spatial decay rate of the Kalman Filter Gain by applying Theorem 5 (Paley-Wiener) to our problem set-up. We start by analyzing the decay of L_0 . Define the extension of (24) by:

$$\hat{L}_0^{\text{ext}}(z; \pi_e) = \pi_e z^2 + \sqrt{\pi_e^2 z^4 - z^2 + 1}, \quad z \in \mathbb{C}. \quad (28)$$

The region of the complex plane in which (28) is analytic is determined by its branch points. These are given by the branch points of the square root component, namely $|z_\infty| \rightarrow \infty$ and the roots

$$\pi_e^2 z^4 - z^2 + 1 = 0. \quad (29)$$

The branch points z_i ($i \in \{1, 2, 3, 4\}$) from (29) are parametrized by π_e and given by:

$$z_i(\pi_e) = \begin{cases} \pm \frac{1}{\sqrt{2\pi_e}} \sqrt{1 \pm \sqrt{1 - 4\pi_e^2}} & \text{if } 0 < \pi_e < \frac{1}{2}, \\ \frac{1}{\sqrt{2\pi_e}} \left(\pm \sqrt{\pi_e + \frac{1}{2}} \pm i \sqrt{\pi_e - \frac{1}{2}} \right) & \text{if } \pi_e > \frac{1}{2}. \end{cases} \quad (30)$$

The z_i 's are real and distinct for $0 < \pi_e < 1/2$ and for $\pi_e > 1/2$ they are complex conjugates. $\pi_e^* = 1/2$ is a critical value at which branch points collapse pairwise in the real axis (see Fig. 1). Substitution of $\pi_e^* = 1/2$ in (24) yields $\hat{L}_0(\kappa; \pi_e^*) = 1$. That is, at π_e^* the convolution kernel of the Kalman Filter Gain is a Dirac δ distribution and the Kalman filter is *completely decentralized*: at a particular spatial location χ , only the measurement from χ is required to implement the filter. The widest analyticity strip Γ_β (see Theorem 5) that can be defined for the extension (28) corresponds to $\beta :=$

$\min_{i \in \{1,2,3,4\}} |\Re[z_i(\pi_e)]|$. Defining β as the decay rate of L_0 yields (27). A similar calculation reveals that the extension of the remaining component of the dimensionless Kalman Filter Gain, namely $\sqrt{2\pi_e L_0}$, has the same analyticity strip of L_0 . Thus, we refer to β as the universal spatial decay rate of the Kalman Filter Gain. \square

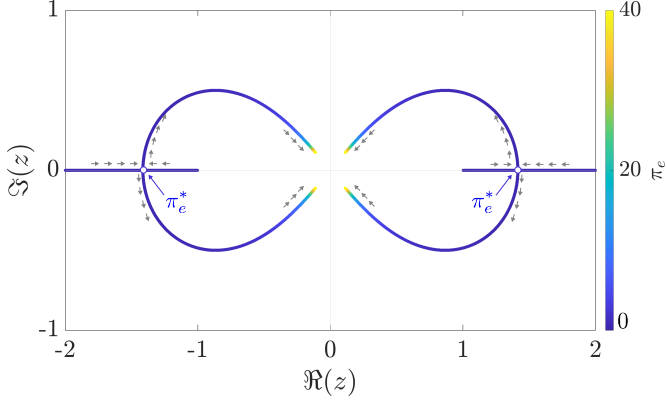


Fig. 1. Trajectories of the branch points (30) in the complex plane as π_e varies – Branch Point Locus, as introduced in Arbelaiz et al. (2020). Trajectories are color-coded according to the value of π_e indicated in the colorbar. Arrows indicate the direction of increasing π_e to guide the eye.

We highlight the peculiar structure of the Kalman Filter Gain. We note that

$$\lim_{|\kappa| \rightarrow \infty} \hat{L}_0(\kappa) = \lim_{|\kappa| \rightarrow \infty} \frac{\kappa^2 + 1}{\pi_e \kappa^2 + \sqrt{\pi_e^2 \kappa^4 + \kappa^2 + 1}} = \frac{1}{2\pi_e}, \quad (31)$$

which shows that the kernel of the Kalman Gain in space consists of a superposition of a Dirac δ distribution with an exponentially decaying component (see Fig. 2), the Fourier transform of the latter being

$$\hat{L}_0^{\text{exp}} = \frac{\pi_e \kappa^2 + 2\pi_e - \sqrt{\pi_e^2 \kappa^4 + \kappa^2 + 1}}{2\pi_e(\pi_e \kappa^2 + \sqrt{\pi_e^2 \kappa^4 + \kappa^2 + 1})} \begin{cases} < 0 & \text{if } \pi_e < 1/2, \\ = 0 & \text{if } \pi_e^* = 1/2, \\ > 0 & \text{if } \pi_e > 1/2. \end{cases}$$

At the critical value $\pi_e^* = 1/2$, the exponentially decaying component of the convolution kernel flips sign and has zero amplitude (see Fig. 2). Hence, only the delta component remains, which explains the emergence of a completely decentralized Kalman Filter.

5. DISCUSSION, CONCLUSION & FUTURE WORK

Discussion. At the critical value π_e^* the Kalman Filter is completely decentralized, that is, the quality of the state estimate does not improve by accessing neighboring measurements. A similar phenomenon was reported in Kalman filters for diffusion processes with spatially correlated measurement noise (see Arbelaiz et al., 2020, Section IV.A). Indeed, in our problem set-up the Sobolev weight $w(\alpha; k)$ as defined in (7) can be effectively thought of as spatially correlated measurement noise, with correlation length α . To see this, define the effective power spectral density of the measurement noise

$$\hat{R}_{\mathbf{n}, \kappa}^{\text{eff}} := \hat{R}_{\mathbf{n}, \kappa} / w_{\kappa}. \quad (32)$$

$\hat{R}_{\mathbf{n}, \kappa}^{\text{eff}}$ is a spatially low-pass filtered version of $\hat{R}_{\mathbf{n}, \kappa}$ (i.e., \mathbf{n}_{eff} is exponentially autocorrelated in space). Substitution of (32) in the ARE (35) yields

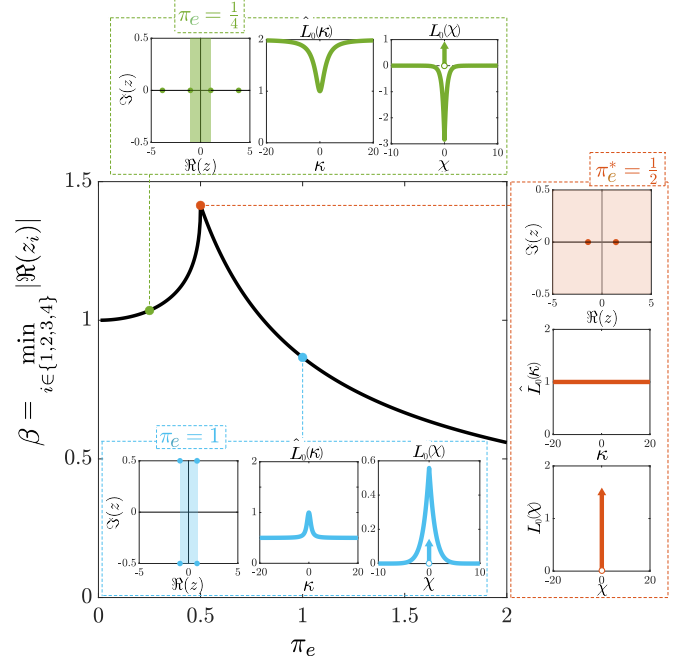


Fig. 2. (Central panel) Universal decay rate β of the Kalman Gain as a function of π_e as defined in (27). (Side panels) Branch points & analyticity strip, $\hat{L}_0(\kappa)$ as given in (24), and $L_0(\chi)$ for different values of π_e : (top, in green) $\pi_e = 1/4$; (right, in orange) $\pi_e^* = 1/2$; (bottom, in blue) $\pi_e = 1$.

$$\hat{A}_{\kappa} \hat{\Pi}_{\kappa} + \hat{\Pi}_{\kappa} \hat{A}_{\kappa}^* + \hat{B}_{\kappa} \hat{B}_{\kappa}^* - \hat{\Pi}_{\kappa} \hat{C}_{\kappa}^* (\hat{R}_{\mathbf{n}, \kappa}^{\text{eff}})^{-1} \hat{C}_{\kappa} \hat{\Pi}_{\kappa} = 0,$$

with the corresponding Kalman Gain $\hat{L}_{\kappa} = \hat{\Pi}_{\kappa} \hat{C}_{\kappa}^* (\hat{R}_{\mathbf{n}, \kappa}^{\text{eff}})^{-1}$.

We remark that the choice of state and realization (11) for the wave equation is not unique. In applications such as H_2 output feedback control then, there may be advantages to choosing one realization over another for the state estimate, as long as the filtering problem remains well-posed. Choosing a realization for a “more localized” estimator could lead to more localization of the overall control policy.

Conclusion. We analyzed the Kalman Filter for an elastic wave equation with partial state observations over the real line. Using dimensional analysis and a Paley-Wiener theorem, we characterized the universal asymptotic spatial decay rate of the Kalman Gain, proving that it is determined by a unique dimensionless parameter defined from the physical parameters of the plant. We found that certain parameter regimes yield a totally decentralized Kalman Filter.

Future Work. Ongoing work includes the characterization of the Kalman Filter performance as a function of the spatial localization of the Kalman Gain, and in particular analysis of performance corresponding to the critical parameter value π_e^* . We also aim to impose strict decentralization constraints using techniques from Arbelaiz et al. (2021) or perhaps convex relaxations inspired by (Jensen, 2020, Ch. 5) to design Kalman Filters with information constraints. Following this alternate approach and characterizing the performance gap between these two problems is part of our ongoing work. Study of the finite spatial domain setting as a step towards analysis of crawling locomotion and other applications is another line of future work.

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APPENDIX: PROOF OF PROPOSITION 7

Spatial invariance of A, B, C imply (see Bamieh et al., 2002) that the solution P of the ARE (22) is a spatially-invariant operator and thus, so is the Kalman Filter Gain L (20). By Propositions 2 and 6 the ARE (22) may be written in the spatial frequency domain:

$$0 = \langle \hat{P}h, \widehat{A^\dagger} \hat{\ell} \rangle + \langle \widehat{A^\dagger} h, \hat{P} \hat{\ell} \rangle + \langle \widehat{B} \widehat{B^\dagger} h, \hat{\ell} \rangle - \langle \widehat{P} \widehat{C^\dagger} \widehat{C} P h, \hat{\ell} \rangle, \quad (33)$$

for all $\hat{\ell}, h \in \mathcal{D}(\widehat{A^\dagger})$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on the weighted L^2 space L_W with spatial frequency weighting function $W_\kappa = W(\kappa) = \begin{bmatrix} 1 + \kappa^2 & 0 \\ 0 & 1 \end{bmatrix}$. Expanding out these inner products, (33) becomes

$$0 = \int_{\kappa \in \mathbb{R}} \hat{\ell}_\kappa^* W_\kappa F(\kappa) \hat{h}_\kappa d\kappa, \quad \text{for all } \hat{\ell}, h \in \mathcal{D}(\widehat{A^\dagger}), \quad (34)$$

where $F(\kappa) := \hat{A}_\kappa \hat{P}_\kappa + \hat{P}_\kappa \hat{A}_\kappa^\dagger + \hat{B}_\kappa \hat{B}_\kappa^\dagger - \hat{P}_\kappa \hat{C}_\kappa^\dagger \hat{C}_\kappa \hat{P}_\kappa$. As $\mathcal{D}(\hat{A}_\kappa^\dagger)$ is dense in \mathcal{X}' and W_κ is invertible for all κ , (34) holds if and only if $F(\kappa) \equiv 0$. Computing adjoint symbols using Proposition 6 and defining a new variable $\hat{\Pi}_\kappa = \hat{\Pi}_\kappa^* = \hat{P}_\kappa W_\kappa^{-1} \succ 0$, $F(\kappa) \equiv 0$ is equivalent to:

$$\hat{A}_\kappa \hat{\Pi}_\kappa + \hat{\Pi}_\kappa \hat{A}_\kappa^* + \hat{B}_\kappa \hat{B}_\kappa^* - \hat{\Pi}_\kappa \hat{C}_\kappa^* (1 + \kappa^2) \hat{C}_\kappa \hat{\Pi}_\kappa = 0. \quad (35)$$

Thus the operator Riccati equation (33) holds if and only if the family of matrix Riccati equations (35) parameterized by κ hold. Solving explicitly for $\hat{\Pi}_\kappa = \begin{bmatrix} \hat{\Pi}_1 & \hat{\Pi}_0 \\ \hat{\Pi}_0 & \hat{\Pi}_2 \end{bmatrix}$, we obtain

$$\hat{\Pi}_0(\kappa) = \frac{-\pi_e^2 \kappa^2}{1 + \kappa^2} + \sqrt{\frac{\pi_e^4 \kappa^4}{(1 + \kappa^2)^2} + \frac{\pi_e^2}{1 + \kappa^2}},$$

$$\hat{\Pi}_1(\kappa) = \sqrt{\frac{2\pi_e^2}{1 + \kappa^2} \Pi_0}, \quad \hat{\Pi}_2 = \hat{\Pi}_1 \left(\kappa^2 + \hat{\Pi}_0 \frac{1 + \kappa^2}{\pi_e^2} \right).$$

The Fourier symbol of the corresponding Kalman gain is $\hat{L}_\kappa = \hat{\Pi}_\kappa \hat{C}_\kappa^* (1 + \kappa^2)$, which can be written as (23).