Worst-Case Algorithm Analysis

Bridging Discrete and Continuous Time

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Algorithm analysis and design

- Many sources of inspiration:
  dynamical systems, physics, mechanics, control systems.

- Iterative algorithms are fundamentally discrete, but useful interpretations and insight can be gained from examining the continuous-time limit.

Key take-aways

1. Many concepts from dynamical systems and controls have analogous discrete-time and continuous-time versions.

2. When seeking *robust* versions of these concepts, discrete and continuous time may not be equivalent.

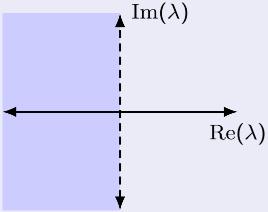
3. Worst-case algorithm analysis is a *robust control problem*.

Stability

continuous time

$$\dot{x}(t) = A\,x(t)$$

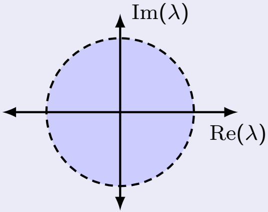
- Asymptotically stable if and only if $\mathrm{Re}(\lambda) < 0$ for all eigenvalues of A.
- In this case, $\|x(t)\| \to 0$ at a linear (exponential) rate.



discrete time

$$x_{k+1} = A\,x_k$$

- Asymptotically stable if and only if $|\lambda| < 1$ for all eigenvalues of A.
- In this case, $\|x_k\| \to 0$ at a linear (exponential) rate.

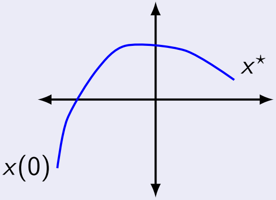


Reachability

continuous time

$$\dot{x}(t) = Ax(t) + Bu(t)$$

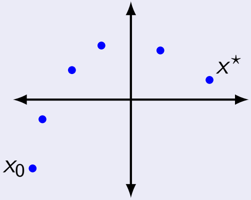
Any state $x^\star$ can be reached in a finite time T with suitable choice of input $u(t)$ if and only if the controllability matrix $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ has full row rank.



discrete time

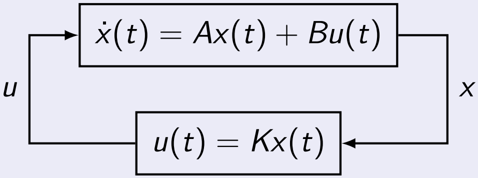
$$x_{k+1} = Ax_k + Bu_k$$

Any state $x^\star$ can be reached in finite time with suitable choice of input $\{u_k\}$ if and only if the controllability matrix $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ has full row rank.



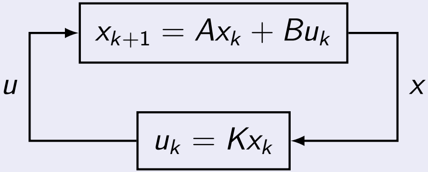
Pole placement

continuous time



A state feedback matrix K can be used to place poles of $\dot{x}(t) = (A + BK)x(t)$ *anywhere* iff (A, B) is controllable.

discrete time



A state feedback matrix K can be used to place poles of $x_{k+1} = (A + BK)x_k$ *anywhere* iff (A, B) is controllable.

Lyapunov functions

Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is a positive definite function.

continuous time

$$\dot{x}(t) = f(x(t))$$

If $\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) < 0$ then the origin is a stable equilibrium point.

discrete time

$$x_{k+1} = f(x_k)$$

If $V(x_{k+1}) - V(x_k) < 0$ then the origin is a stable equilibrium point.

In both cases, we call V a *Lyapunov function*.

Lyapunov functions for linear systems

continuous time

$$\dot{x}(t) = A\,x(t)$$

- System is stable iff there exists a quadratic Lyapunov function (of the form $V(x) = x^\mathsf{T} P x$)
- This happens iff there is a $P \succ 0$ satisfying $A^\mathsf{T} P + PA \prec 0$.

discrete time

$$x_{k+1} = Ax_k$$

- System is stable iff there exists a quadratic Lyapunov function. (of the form $V(x) = x^\mathsf{T} P x$)
- This happens iff there is a $P \succ 0$ satisfying $A^\mathsf{T} PA - P \prec 0$.

If dynamics are *unknown* or *uncertain*, it is a **robust control** problem.

- Robust versions of stability, reachability, pole placement, etc.
- In these cases, discrete-time and continuous-time versions are often *not* equivalent!

Interval polynomials

Suppose the characteristic polynomial of A is

$$q_0 + q_1\lambda + \cdots + q_{n-1}\lambda^{n-1} + q_n\lambda^n$$

But there is bounded uncertainty in the coefficients: $q_i \in [a_i, b_i]$.
This scenario can arise when identifying/learning a system from data.

Question: Are all such polynomials stable?

Kharitonov's theorem for continuous-time dynamics

Theorem (Kharitonov, 1978)

Every polynomial of the form $q_0 + q_1\lambda + \cdots + q_n\lambda^n$ with $a_i \le q_i \le b_i$ and $0 \notin [a_n, b_n]$ is *continuous-time stable* (all roots satisfy $\mathrm{Re}(\lambda) < 0$) if and only if the following four polynomials are continuous-time stable:

$$a_0 + a_1\lambda + b_2\lambda^2 + b_3\lambda^3 + a_4\lambda^4 + a_5\lambda^5 + \cdots$$
$$b_0 + a_1\lambda + a_2\lambda^2 + b_3\lambda^3 + b_4\lambda^4 + a_5\lambda^5 + \cdots$$
$$b_0 + b_1\lambda + a_2\lambda^2 + a_3\lambda^3 + b_4\lambda^4 + b_5\lambda^5 + \cdots$$
$$a_0 + b_1\lambda + b_2\lambda^2 + a_3\lambda^3 + a_4\lambda^4 + b_5\lambda^5 + \cdots$$

Kharitonov's theorem for discrete-time dynamics?

There is no analogous result for discrete-time stability! Closest version:

Theorem (Hollot & Bartlett, 1986)

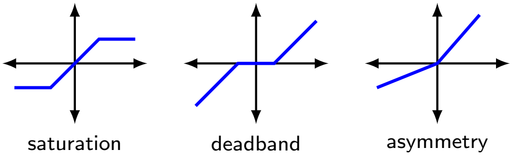
Suppose $a_i = b_i$ for $i > \lfloor \frac{n}{2} \rfloor$. Every polynomial of the form $q_0 + q_1\lambda + \cdots + q_n\lambda^n$ with $a_i \le q_i \le b_i$ is *discrete-time stable* (all roots satisfy $|\lambda| < 1$) if and only if each of the following polynomials are stable

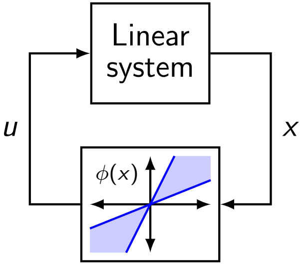
$$q_0 + q_1\lambda + \cdots + q_n\lambda^n \quad \text{where } q_i \in \{a_i, b_i\} \text{ for all } i.$$

- Requires stronger assumptions than continuous-time version.
- Number of polynomials to check is $O(2^{\lfloor n/2 \rfloor})$.
- Other variants of this result exist.

Lur'e problem (1944)

- Many electical/mechanical systems are *mostly* linear but contain simple nonlinearities. Examples of $\phi(x)$:



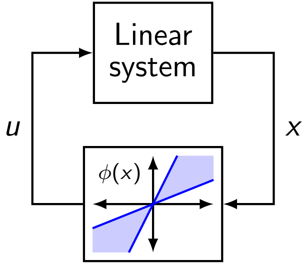


- Define a sector $a < \phi'(x) < b$ that contains the nonlinearity and look for *robust stability*.

Lur'e problem: What is required of the linear system so that the closed-loop system is stable *for all possible [slope-restricted]* ϕ.

Kalman conjecture (1957)

Suppose we verify asymptotic stability for all *linear* $\phi(x) = kx$ with $a < k < b$.
Then we have asymptotic stability for all *slope-restricted* ϕ with $a < \phi'(x) < b$.



This is false in general!

- Continuous-time: true for $n \leq 3$, counterexamples exist for $n \geq 4$. (Fitts, 1966).
- Discrete-time: true for $n = 1$, counterexamples exist for $n \geq 2$.

Optimization algorithms as robust control problems

Key fact

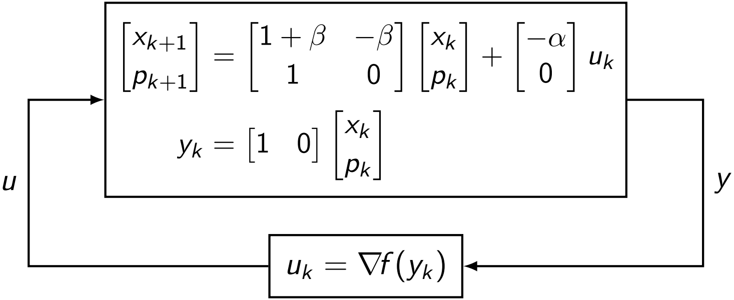
If $f \in C^2(\mathbb{R}^n)$ is m-strongly convex and has L-Lipschitz gradients, then ∇f is *slope-restricted*. In particular, $mI \leq \nabla^2 f(x) \leq LI$ for all x.

- "m-strongly convex with L-Lipschitz gradients" is well studied in optimization.
- Slope-restricted nonlinearities are well-studied in robust control.
- This is not a coincidence. . .

Heavy ball method (Polyak, 1987)

$$x_{k+1} = x_k - \alpha \nabla f(x_k) + \beta(x_k - x_{k-1}), \qquad \text{with: } \alpha = \frac{4}{(\sqrt{L}+\sqrt{m})^2},\ \beta = \left(\frac{\sqrt{L}-\sqrt{m}}{\sqrt{L}+\sqrt{m}}\right)^2$$

Becomes a Lur'e problem if we define $u_k := \nabla f(x_k)$ and $p_k := x_{k-1}$

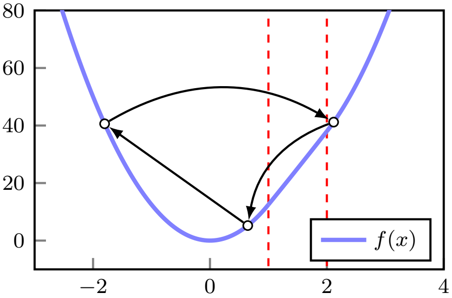


Heavy ball is not globally stable!

Counterexample with $L/m = 25$:

$$f(x) = \begin{cases} \frac{25}{2}x^2 & x < 1 \\ \frac{1}{2}x^2 + 24x - 12 & 1 \le x < 2 \\ \frac{25}{2}x^2 - 24x + 36 & x \ge 2 \end{cases}$$

Initialize heavy ball with $x_0 = x_1 \in [3.07, 3.46]$.



- Iterates converge to a stable limit cycle.
- Simple counterexample to the (discrete-time) Kalman conjecture.
- The corresponding second-order ODE is robustly asymptotically stable.

Key take-aways

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2. When seeking *robust* versions of these concepts, discrete and continuous time may not be equivalent.

3. Worst-case algorithm analysis is a *robust control problem*.

Thank you!

- Optimization algorithms through the lens of robust control:

  L. Lessard, B. Recht, A. Packard. "Analysis and Design of Optimization Algorithms via Integral Quadratic Constraints". (SIOPT'16)

- `https://laurentlessard.com`