Integral Quadratic Constraints

Exact Convergence Rates and Worst-Case Trajectories

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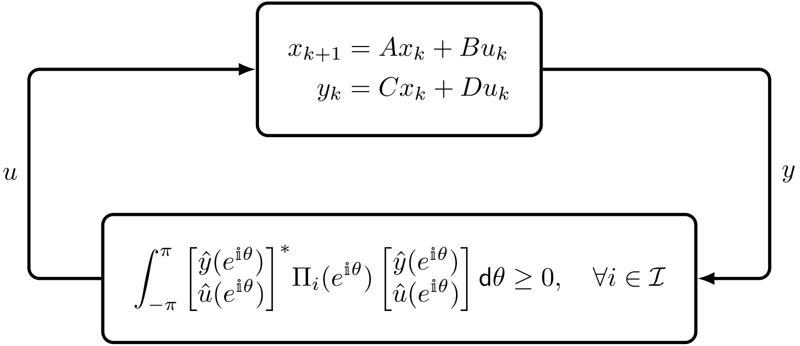
Overview



Problem

Efficiently determine if the system is robustly stable.

- if stable, provide a *Lyapunov function*
- otherwise, construct an *unstable trajectory*

Literature

- Megretski and Rantzer (1997)
  - frequency-domain
  - soft IQCs
  - use KYP lemma to obtain an LMI with *symmetric* P

> *Remark 4:* It is important to note that if $\tau\Delta$ satisfies several IQC's, defined by $\Pi_1, \cdots, \Pi_n$, then a sufficient condition for stability is the existence of $x_1, \cdots, x_n \geq 0$ such that (9) holds for $\Pi = x_1\Pi_1 + \cdots + x_n\Pi_n$. Hence, the more IQC's that can be verified for Δ, the better. Furthermore, the condition is necessary in the following sense. If it fails for all $x_i \geq 0$, then (5) fails for some signals f, v, w with $v = Gw + f$ and v, w satisfying all the IQC's [17], [18].

- Seiler (2015)
  - time-domain
  - hard IQCs
  - dissipation LMI with *positive semidefinite* P

Contribution

Prove that the IQC theorem is tight by constructing worst-case trajectories.

Review: autonomous LTI systems

$$x_{k+1} = Ax_k$$

	Stability	**Instability**
Certificate	there exists a quadratic Lyapunov function	there exists an unstable trajectory
LMI	there exists $P \succ 0$ such that $A^\mathsf{T} P A - P \prec 0$	there exists nonzero $Q \succeq 0$ such that $AQA^\mathsf{T} - Q \succeq 0$
Spectral radius	$\rho(A) < 1$	$\rho(A) \geq 1$

see Balakrishnan and Vandenberghe (2003) for an overview on alternatives for problems in control

Spectral radius

$$\begin{aligned} \rho(A) = \underset{\rho,\, P}{\text{infimum}} \quad & \rho \\ \text{subject to} \quad & 0 \succeq A^{\mathsf{T}} P A - \rho^2 P \\ & \rho > 0 \\ & P \succ 0 \end{aligned}$$

- Checking feasibility for fixed ρ is an LMI.
- There exists a feasible point for any $\rho > \rho(A)$.
- There does *not* exist a feasible point for any $\rho < \rho(A)$.

We can efficiently compute the spectral radius by bisecting over ρ.

Stability

$\rho(A) < 1$	$\Longrightarrow$	asymptotically stable
$\rho(A) \leq 1$ and optimum attained	$\Longrightarrow$	bounded

The optimum may *not* be attained.

- Consider the example $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
- The spectral radius is $\rho(A) = 1$, but there does *not* exist $P \succ 0$ such that $A^\mathsf{T} P A - P \preceq 0$.
- The state grows unbounded with initial condition $x_0 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$.

Worst-case trajectory

When $\rho(A) = 1$, we can construct an unstable trajectory as follows:

(1) Find nonzero $Q \succeq 0$ such that $AQA^\mathsf{T} - Q = 0$.

(2) Factor $Q = XX^\mathsf{T}$.

(3) Find an orthonormal matrix F such that $AX = XF$.

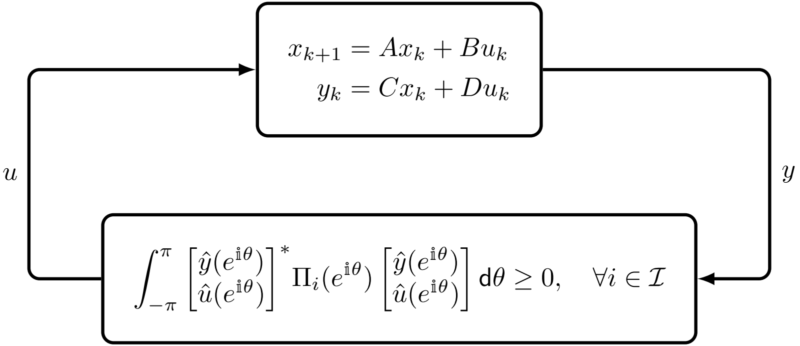
(4) Then for any nonzero vector v, a worst-case trajectory is

$$x_k = XF^k v.$$

Note that this is a valid trajectory since

$$x_{k+1} = XF^{k+1}v = AXF^k v = Ax_k.$$

LTI system subject to IQCs



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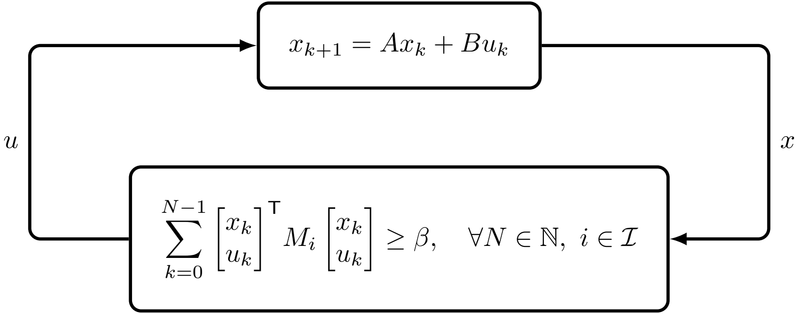
From dynamic to static IQCs

The frequency-domain IQCs may have *dynamics*. Instead, we can

- factor each multiplier $\Pi_i(z)$,
- combine the dynamic parts with the LTI system, and
- use Parseval's theorem to produce *static* time-domain IQCs.



Note: A, B, and x_k have been modified to include the IQC dynamics

Generalized spectral radius

$$
\begin{aligned}
\rho(A, B, \mathcal{M}) = \underset{\rho,\, P,\, \lambda_i}{\text{infimum}} \quad & \rho \\
\text{subject to} \quad & 0 \succeq \begin{bmatrix} A^\mathsf{T} P A - \rho^2 P & A^\mathsf{T} P B \\ B^\mathsf{T} P A & B^\mathsf{T} P B \end{bmatrix} + \sum_{i \in \mathcal{I}} \lambda_i \, M_i \\
& \rho > 0 \\
& P \succ 0 \\
& \lambda_i \geq 0 \quad \text{for all } i \in \mathcal{I}
\end{aligned}
$$

- Generalizes the spectral radius of a matrix.
- Efficiently computable by bisecting over ρ.
- The optimum may not be attained.
- For fixed ρ, this is similar to the LMI obtained from applying the KYP lemma to the IQC theorem, but with *positive definite* P.

Robust stability

Theorem

$\rho(A, B, \mathcal{M}) < 1 \implies$ robustly asymptotically stable

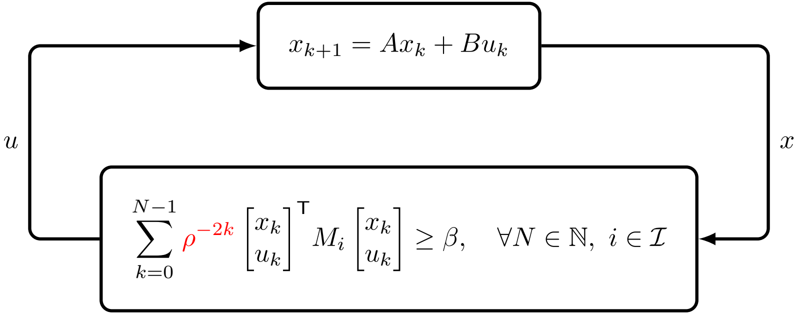
$\rho(A, B, \mathcal{M}) \leq 1$ and opt attained $\implies$ robustly bounded

- Result is similar to the autonomous case.
- Proof uses the Lyapunov function

$$V_k = x_k^\mathsf{T} P x_k + \sum_{j=0}^{k-1} \begin{bmatrix} x_j \\ u_j \end{bmatrix}^\mathsf{T} \left(\sum_{i \in \mathcal{I}} \lambda_i \, M_i \right) \begin{bmatrix} x_j \\ u_j \end{bmatrix}.$$

- Straighforward generalization to *robust exponential stability*.

Robust exponential stability



Corollary

Let $\rho = \rho(A, B, \mathcal{M})$, and suppose the optimum is attained. Then there exists a constant $c > 0$ such that $\|x_k\| \leq c\,\rho^k\,\|x_0\|$ for all k.

Worst-case trajectory

Lemma

Suppose $\rho(A, B, \mathcal{M}) = 1$ and B is full column rank. Then there exist matrices X, U, and F with X nonzero and F orthonormal such that

$$AX + BU = XF$$

and

$$\operatorname{trace}\left(\begin{bmatrix} X \\ U \end{bmatrix}^\mathsf{T} M_i \begin{bmatrix} X \\ U \end{bmatrix}\right) \geq 0 \quad \text{for all } i \in \mathcal{I}.$$

- Result is similar to the autonomous case.
- Proof uses SDP duality and linear algebra.
- If B is not full column rank, then combine inputs to make it full rank.

Worst-case trajectory

Theorem

Suppose $\rho(A, B, \mathcal{M}) = 1$, B is full column rank, and there exists a vector v that satisfies a *technical condition* for some (X, U, F) from the previous lemma. Then

$$\begin{bmatrix} x_k \\ u_k \end{bmatrix} = \begin{bmatrix} X \\ U \end{bmatrix} F^k v$$

is a trajectory that is *not* asymptotically stable.

- In some cases, the trajectory also satisfies the *hard* or *pointwise* IQCs.
- **Static state feedback:** If X is full column rank, then

$$u_k = (U X^\dagger)\, x_k.$$

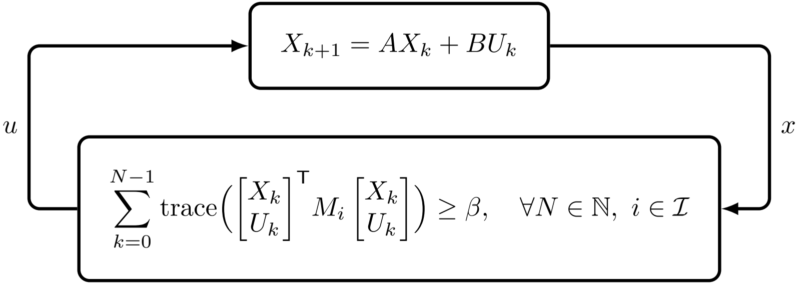
- In contrast to the autonomous case, we require an additional *technical condition* (more on next slide).

Technical condition

Even without the technical condition, the trajectory

$$\begin{bmatrix} X_k \\ U_k \end{bmatrix} = \begin{bmatrix} X \\ U \end{bmatrix} F^k$$

satisfies the following dynamics:



The technical condition ensures that we can construct a *vector* x_k from the *matrix* X_k.

Summary

- We efficiently characterize robust stability of an LTI system subject to a set of integral quadratic constraints.
  - If robustly stable, we provide a *Lyapunov function* of the form

$$V_k = x_k^\mathsf{T} P x_k + \sum_{j=0}^{k-1} \begin{bmatrix} x_j \\ u_j \end{bmatrix}^\mathsf{T} \left(\sum_{i \in \mathcal{I}} \lambda_i \, M_i \right) \begin{bmatrix} x_j \\ u_j \end{bmatrix}.$$

  - Otherwise, we construct a *worst-case trajectory* of the form

$$\begin{bmatrix} x_k \\ u_k \end{bmatrix} = \begin{bmatrix} X \\ U \end{bmatrix} F^k v.$$

- Generalizes *linear-quadratic Lyapunov theory* for autonomous systems.
- Provides a constructive proof of the worst-case trajectory mentioned in Remark 4 of Megretski and Rantzer (1997).