Unified necessary and sufficient conditions for the robust stability of interconnected sector-bounded systems

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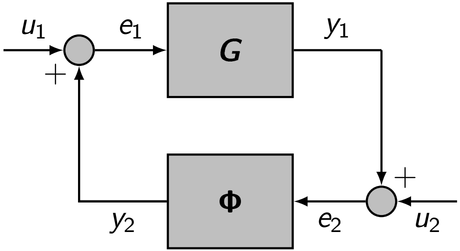
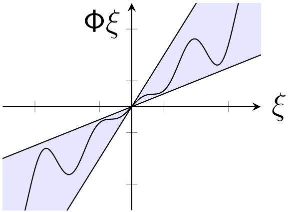
Robust stability of interconnected systems



- G is known, linear, and time-invariant.
- Φ is a sector-bounded nonlinearity.



When can we guarantee that the system will be BIBO stable?
(bounded $\|u\| \implies$ bounded $\|y\|$)

Many existing results

- Circle criterion and conicity (Brockett 1964, Zames 1964, Sandberg 1964).
- The passivity theorem (Zames 1968).
- The small-gain theorem (Zames 1968).
- Extended conicity theorem (Bridgeman & Forbes 2016).
- Dissipativity theory (Willems 1972).
- Graph separation results (Safonov 1980, Teel 1996).
- Integral Quadratic Constraints (IQCs) (Megretski & Rantzer 1997).

A plethora of reformulations are also present
in every nonlinear/robust controls textbook.

Examples from the literature

58 Theorem *Consider the feedback system of Figure 6.2. Suppose there exist constants* $\varepsilon_i, \delta_i, i = 1, 2$, *such that*

59 $<x, G_i x>_T \geq \varepsilon_i \|x\|_{T2}^2 + \delta_i \|G_i x\|_{T2}^2, \forall T \geq 0, \forall x \in L_{2e}, i = 1, 2.$

Then the system is L_2*-stable wb if*

60 $\delta_1 + \varepsilon_2 > 0, \delta_2 + \varepsilon_1 > 0.$

A Circle Theorem. *Suppose that*

(I) $\boldsymbol{N}$ *is a relation in* $\mathfrak{R}_0$, *(incrementally) inside the sector* $\{\alpha+\Delta, \beta-\Delta\}$, *where* $\beta > 0$.

(II) $\boldsymbol{H}$ *is an operator in* $\mathfrak{L}$, *which satisfies the circle conditions for the sector* $\{\alpha, \beta\}$ *with offset* δ.

(III) δ *and* Δ *are non-negative constants, at least one of which is greater than zero.*

Then the closed-loop operators $\boldsymbol{E}_1$ *and* $\boldsymbol{E}_2$ *are* L_2*-bounded (*L_2*-continuous).*

Sufficient results: Passivity theorem and Circle theorem (Vidyasagar).

THEOREM 2a: *[In eqs. (1)–(2)] Let* $\boldsymbol{H}_1$ *and* $\boldsymbol{H}_2$ *be conic relations. Let* Δ *and* δ *be constants, of which one is strictly positive and one is zero. Suppose that*

(I) $-\boldsymbol{H}_2$ *is inside the sector* $\{a+\Delta,\ b-\Delta\}$ *where* $b>0$, *and,*

(II) $\boldsymbol{H}_1$ *satisfies one of the following conditions.*

CASE 1a: *If* $a>0$ *then* $\boldsymbol{H}_1$ *is outside*

$$\left\{-\frac{1}{a}-\delta,\ -\frac{1}{b}+\delta\right\}.$$

CASE 1b: *If* $a<0$ *then* $\boldsymbol{H}_1$ *is inside*

$$\left\{-\frac{1}{b}+\delta,\ -\frac{1}{a}-\delta\right\}.$$

CASE 2: *If* $a=0$ *then*

$$H_1+\left(\frac{1}{b}-\delta\right)I$$

is positive; in addition, if $\Delta=0$ *then* $g(\boldsymbol{H}_1)<\infty$.

Then $\mathbf{E}_1$ *and* $\mathbf{E}_2$ *are bounded.*

Theorem 3.1: Let $\mathbf{d}^{\mathsf{T}}=[\mathbf{d}_1^{\mathsf{T}}\ \ \mathbf{d}_2^{\mathsf{T}}]\in X_e\times X_e$ be an external signal, and $\mathcal{G}_1,\mathcal{G}_2: X_e\mapsto X_e$ a plant and controller in negative feedback as described by (4) and shown in Fig. 1. For the closed-loop system, $\mathcal{G}: X_e\times X_e\mapsto X_e\times X_e$, defined as $\mathbf{y}=\mathcal{G}\mathbf{d}$, the following conditions guarantee I-O stability:

1) If $\mathcal{G}_1\in\text{cone}[a,b]$, $a<b$ and
   - a) $\mathcal{G}_2\in\text{cone}(-(1/b),-(1/a))$ if $ab\le 0$, or
   - b) $\mathcal{G}_2\in\text{excone}(-(1/a),-(1/b))$ if $0\le ab$.

2) If $\mathcal{G}_1\in\text{excone}[a,b]$, $a<b$, and
   - a) $\mathcal{G}_2\in\text{excone}(-(1/b),-(1/a))$ if $ab\le 0$, or
   - b) $\mathcal{G}_2\in\text{cone}(-(1/a),-(1/b))$ if $0\le ab$.

If $\mathcal{G}_1$ is strictly conic, the stability guarantees still hold while allowing $\mathcal{G}_2$ to simply be conic, rather than strictly so.

Sufficient results: Conicity (Zames) and Extended Conicity (Bridgeman & Forbes).

126 Theorem *Consider the system of Figure 6.27, and suppose a, b are two given real numbers with $a < b$. Then the following two statements are equivalent:*

(I) The feedback system is L_2-stable wb for every Φ belonging to the sector (a, b).

(II) The transfer function $\hat{g}$ satisfies one of the following conditions as appropriate: (1) If $ab > 0$, then the Nyquist plot of $\hat{g}(j\omega)$ does not intersect the interior of the disk $D(a, b)$ defined in (39), and encircles the interior of the disk $D(a, b)$ exactly μ_+ times in the counter-clockwise direction, where μ_+ is the number of poles of $\hat{g}$ with positive real part. (2) If $a = 0$, then $\hat{g}$ has no poles with positive real part, and

127 $\operatorname{Re}\hat{g}(j\omega) \geq -\frac{1}{b}, \forall\omega.$

(3) If $ab < 0$, then $\hat{g} \in \hat{\mathbf{A}}$, and the Nyquist plot of $\hat{g}(j\omega)$ lies inside the disk $D(a, b)$ for all ω.

Theorem 3. *Given bounded $\Sigma_1 = G + \Delta$, where G is linear time-invariant and Δ is linear passive, then there exists $\gamma > 0$ such that the closed-loop system $\Sigma_1 \parallel \Sigma_2$ has $\mathbf{L}_2$-gain $\leq \gamma$ for all bounded passive Σ_2 if and only if Σ_1 is strictly passive.*

Theorem 4. *Given bounded $\widetilde{\Sigma}_1 = G + \Delta$, where G is linear time-invariant and Δ is linear passive, then there exists $\gamma > 0$ such that the closed-loop system $\widetilde{\Sigma}_1 \parallel \widetilde{\Sigma}_2$ has $\mathbf{L}_2$-gain $\leq \gamma$ for all output strictly passive $\widetilde{\Sigma}_2$ if and only if $\widetilde{\Sigma}_1$ is output strictly passive.*

Theorem 5. *Given bounded $\Sigma_1 = G + \Delta$, where G is linear time-invariant and Δ is linear passive, then there exists $\gamma > 0$ such that the closed-loop system $\Sigma_1 \|_{e_2=0} \Sigma_2$ has $\mathbf{L}_2$-gain from e_1 to y_1 less than or equal to γ for all bounded passive Σ_2 if and only if Σ_1 is output strictly passive.*

Necessary and sufficient results:
Circle theorem (Vidyasagar) and Passivity theorems (Khong & van der Schaft).

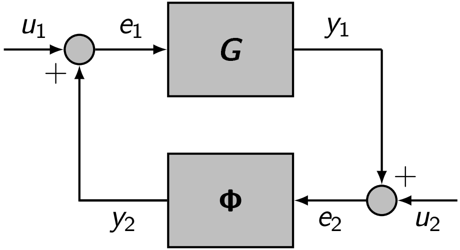
Summary of circle-like results

- **Different assumptions:** linear, causal, bounded, stable, static, well-posed, ...
- **Different spaces:** generalized relations, operators on L_{2e} or ℓ_{2e}, ...
- Sufficient-only vs necessary-and-sufficient results.
- Many special cases.

> - Are all these results really just the same result?
> - What is the simplest version of this result?

Key findings

- Passivity, Small-gain, Conicity, Circle are equivalent via change of variables.
- If G is linear and Φ is allowed to be dynamic, result is necessary and sufficient. Necessity is a consequence of the S-lemma.
- If either condition is relaxed (G is not linear or Φ is static), result is sufficient-only.
- Assumptions needed (causality, stability, boundedness, well-posedness) are consequences of the structure of the spaces used (e.g. L_{2e}).



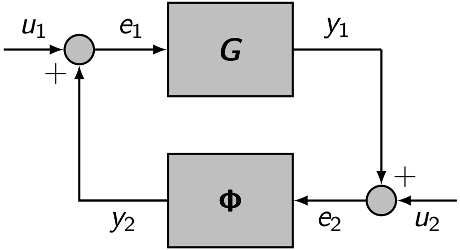
Toward a unified result

What are the minimal ingredients required for a general, unified result?

- **Need** the ability to sum signals and to have an inner product.
- **Need** the ability to impose restrictions (e.g. G and Φ are causal in the L_{2e} case).
- **Want** a parameterization of sectors that encompasses passivity, small-gain, circle without the need for special cases.

Yes, this is possible!

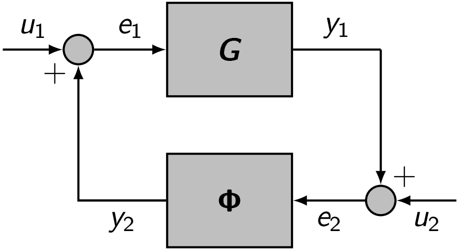
Toward a unified result



$$
\begin{aligned}
e_1 &= u_1 + y_2 \\
(e_2, y_2) &\in \Phi \\
e_2 &= u_2 + y_1 \\
(e_1, y_1) &\in G
\end{aligned}
$$

- $u_1, u_2, e_1, e_2, y_1, y_2 \in \mathcal{X}$. $\mathcal{X}$ is a **semi-inner product space**. Similar to an inner product space but $\langle x, x \rangle = 0$ does not imply that $x = 0$.
- G and Φ are **relations**. They are defined by a set of admissible input-output pairs (they need not be functions!). We call $\mathcal{R}(\mathcal{X})$ the set of all relations on $\mathcal{X}$ and $\mathcal{L} \subseteq \mathcal{R}(\mathcal{X})$ the set of all **linear relations**.

Toward a unified result



$$(u_2 + y_1, y_2) \in \Phi$$
$$(u_1 + y_2, y_1) \in G$$

How do we impose **restrictions**? e.g. "let G be causal operator on L_{2e}"?

- We call a set of relations $\mathcal{C}$ **complete** if for any $x, y \in \mathcal{X}$, there exists a $\Phi \in \mathcal{C}$ such that $(x, y) \in \Phi$.
- We call a set of relations $\mathcal{C}$ **feedback-invariant** if for all $G, \Phi \in \mathcal{C}$, we have $\{(u_i, y_j) \mid u, y \text{ satisfies feedback equations}\} \in \mathcal{C}$

Theorem (Main result)

Let $\mathcal{X}$ be a nontrivial semi-inner product space, let $M = M^{\mathsf{T}} \in \mathbb{R}^{2\times 2}$ be given and let $\mathcal{C} \subseteq \mathcal{R}(\mathcal{X})$ be complete and feedback-invariant. Suppose $G \in \mathcal{L} \cap \mathcal{C}$. The following are equivalent.

1. *There exists $N = N^{\mathsf{T}} \in \mathbb{R}^{2\times 2}$ satisfying $M + N \prec 0$ (positive definite sense) such that G satisfies*

$$\left\langle \begin{bmatrix} G\xi \\ \xi \end{bmatrix}, N \begin{bmatrix} G\xi \\ \xi \end{bmatrix} \right\rangle \geq 0 \qquad \text{for all } \xi \in \mathcal{X}. \quad \text{(Property of } G\text{)}$$

2. *There exists $\gamma > 0$ such that for all $\Phi \in \mathcal{C}$, if*

$$\left\langle \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix}, M \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix} \right\rangle \geq 0 \qquad \text{for all } \xi \in \mathcal{X}, \quad \text{(Property of } \Phi\text{)}$$

   then for all admissible (u, y) in the feedback interconnection of G and Φ, the following holds

$$\|y\| \leq \gamma \|u\|.$$

Example with L_{2e}

- Let $\mathcal{X} = \ell_{2e}(\mathbb{R}^n)$
- Let $\langle x,\, y\rangle_T = \sum_{k=0}^{T} \langle x_k,\, y_k\rangle$ (truncated inner product)
- Let $\mathcal{C}$ be the set of all causal operators. Note that $\mathcal{C}$ is feedback-invariant (interconnections of causal systems are causal).
- If we force G and Φ to be operators (functions) then feedback invariance of $\mathcal{C}$ is equivalent to well-posedness of the interconnection.
- In the main theorem, γ does not depend on $\langle \cdot,\, \cdot \rangle$. It follows that we can apply the result to the truncated inner product for all T and obtain the same γ.

Corollary (ℓ_2 stability)

Let $M = M^\mathsf{T} \in \mathbb{R}^{2\times 2}$ with $M \not\preceq 0$ be given and suppose $G : \ell_{2\mathrm{e}} \to \ell_{2\mathrm{e}}$ is a causal linear operator. The following statements are equivalent:

1. *There exists $N = N^\mathsf{T} \in \mathbb{R}^{2\times 2}$ satisfying $M + N \prec 0$ such that for all $\xi \in \ell_{2\mathrm{e}}$ and $T \geq 0$, G satisfies*
$$\left\langle \begin{bmatrix} G\xi \\ \xi \end{bmatrix}, N \begin{bmatrix} G\xi \\ \xi \end{bmatrix} \right\rangle_T \geq 0.$$

2. *There exists $\gamma > 0$ such that for all causal $\Phi : \ell_{2\mathrm{e}} \to \ell_{2\mathrm{e}}$ where the interconnection of G and Φ is well-posed, if the following statement holds for all $T \geq 0$:*
$$\left\langle \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix}, M \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix} \right\rangle_T \geq 0 \qquad \text{for all } \xi \in \ell_{2\mathrm{e}},$$
then for all admissible (u, y) in the feedback interconnection of G and Φ,
$$\|y\| \leq \gamma \|u\| \quad \text{for all } u \in \ell_2.$$

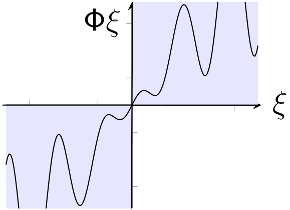
Recovery of necessary and sufficient results

Name of Theorem	M	N	$M+N \prec 0$
Conic sector theorem	$\begin{bmatrix} \frac{-(a+\Delta)(b-\Delta)}{b-a-2\Delta} & \frac{-a-b}{2(b-a-2\Delta)} \\ \frac{-a-b}{2(b-a-2\Delta)} & \frac{-1}{b-a-2\Delta} \end{bmatrix}$	$\begin{bmatrix} \frac{ab}{b-a+2ab\delta} & \frac{a+b}{2(b-a+2ab\delta)} \\ \frac{a+b}{2(b-a+2ab\delta)} & \frac{(1+a\delta)(1-b\delta)}{b-a+2ab\delta} \end{bmatrix}$	$a<b$, and either $\delta=0, \Delta>0$ or $\delta>0, \Delta=0$.
Extended conic sector theorem	$\begin{bmatrix} \frac{(a-\Delta)(b+\Delta)}{b-a+2\Delta} & \frac{a+b}{2(b-a+2\Delta)} \\ \frac{a+b}{2(b-a+2\Delta)} & \frac{1}{b-a+2\Delta} \end{bmatrix}$	$\begin{bmatrix} \frac{-ab}{b-a-2ab\delta} & \frac{-a-b}{2(b-a-2ab\delta)} \\ \frac{-a-b}{2(b-a-2ab\delta)} & \frac{-(1-a\delta)(1+b\delta)}{b-a-2ab\delta} \end{bmatrix}$	Same as above
Extended passivity	$\begin{bmatrix} -\varepsilon_2 & \frac{1}{2} \\ \frac{1}{2} & -\delta_2 \end{bmatrix}$	$\begin{bmatrix} -\delta_1 & -\frac{1}{2} \\ -\frac{1}{2} & -\varepsilon_1 \end{bmatrix}$	$\delta_1+\varepsilon_2>0$ and $\delta_2+\varepsilon_1>0$
Small gain theorem	$\begin{bmatrix} \gamma_2 & 0 \\ 0 & -1/\gamma_2 \end{bmatrix}$	$\begin{bmatrix} -1/\gamma_1 & 0 \\ 0 & \gamma_1 \end{bmatrix}$	$\gamma_1\gamma_2<1$

Recovery of necessary and sufficient results

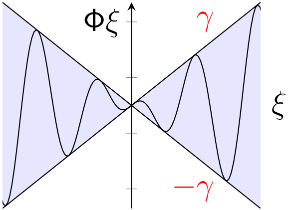
Passivity theorem:

$$M = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$



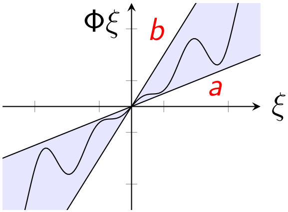
Small-Gain theorem:

$$M = \begin{bmatrix} \gamma & 0 \\ 0 & -\frac{1}{\gamma} \end{bmatrix}$$



Circle criterion:

$$M = \begin{bmatrix} -a & \frac{1}{2}(1+\frac{a}{b}) \\ \frac{1}{2}(1+\frac{a}{b}) & -\frac{1}{b} \end{bmatrix}$$



Weighted Stability Results

Define the set $\ell_2^\rho \subset \ell_2$ of sequences $\{x_k\}$ such that

$$\sum_{k=0}^{\infty} \rho^{-2k} \|x_k\|^2 < \infty$$

For any fixed (ρ, T), define the semi-inner product

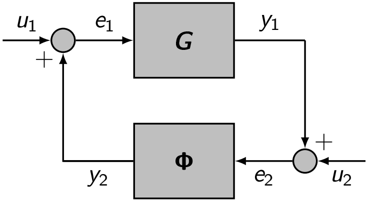
$$\langle x, y \rangle_{\rho,T} := \sum_{k=0}^{T} \rho^{-2k} \langle x_k, y_k \rangle$$

Corollary (Weighted stability)

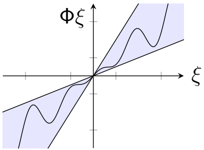
Let $M = M^{\mathsf{T}} \in \mathbb{R}^{2\times 2}$ *with* $M \not\preceq 0$ *and* $\rho \in (0,1]$ *be given. Suppose* $G : \ell_{2\mathrm{e}} \to \ell_{2\mathrm{e}}$ *is causal and linear. The following are equivalent.*

1. *There exists* $N = N^{\mathsf{T}} \in \mathbb{R}^{2\times 2}$ *satisfying* $M + N \prec 0$ *such that for all* $\xi \in \ell_{2\mathrm{e}}^{\rho}$ *and* $T \geq 0$, G *satisfies*
$$\left\langle \begin{bmatrix} G\xi \\ \xi \end{bmatrix}, N \begin{bmatrix} G\xi \\ \xi \end{bmatrix} \right\rangle_{\rho,T} \geq 0.$$
2. *There exists* $\gamma > 0$ *such that for all causal* $\Phi : \ell_{2\mathrm{e}}^{\rho} \to \ell_{2\mathrm{e}}^{\rho}$ *where the interconnection of* G *and* Φ *is well-posed, if the following condition holds for all* $T \geq 0$
$$\left\langle \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix}, M \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix} \right\rangle_{\rho,T} \geq 0 \qquad \text{for all } \xi \in \ell_{2\mathrm{e}}^{\rho},$$
then for all admissible (u, y) *in the feedback interconnection of* G *and* Φ,
$$\|y\|_{\rho} \leq \gamma \|u\|_{\rho} \quad \text{for all } u \in \ell_2^{\rho}.$$

Summary



- G is known, linear, and time-invariant.
- Φ is a sector-bounded nonlinearity.



- General necessary-and-sufficient robust stability result.
- Unifies and generalizes many existing results while using minimal assumptions and no special cases.
- Can be extended to obtain new results by varying the choice of $\mathcal{X}$, $\mathcal{C}$, $\langle\cdot,\cdot\rangle$.

Thank you