State-space solution to a minimum-entropy $\mathcal{H}_\infty$ control problem with nested information

Laurent Lessard

University of California, Berkeley

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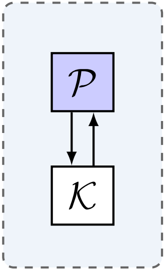
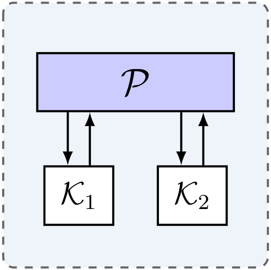
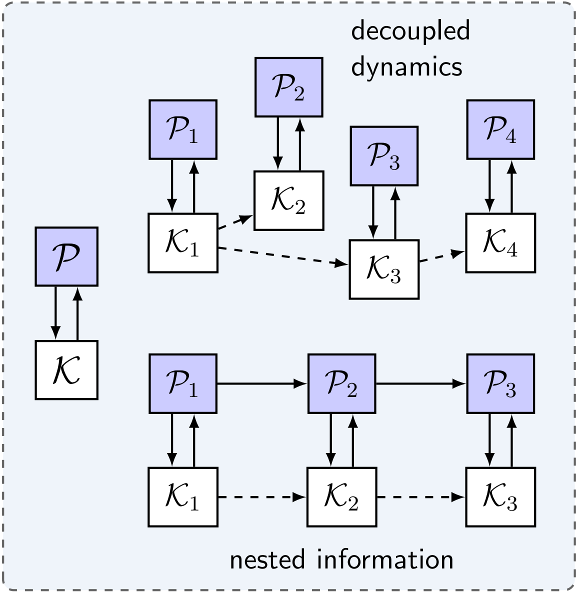


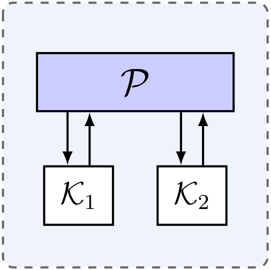
Tractable



Intractable

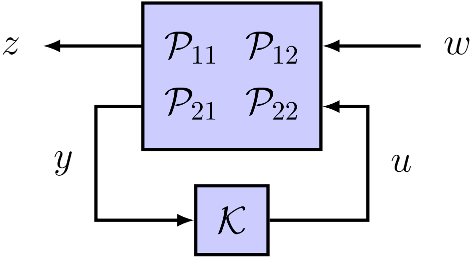


Tractable



Intractable

Classical (centralized) control

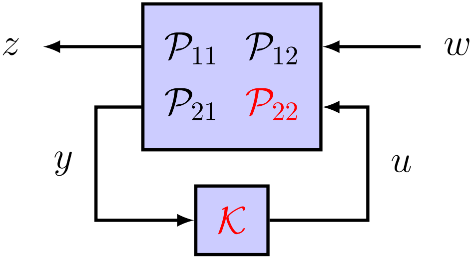


- $\mathcal{P}_{ij}$ and $\mathcal{K}$ are matrices of proper rational transfer functions.
- Closed-loop map: $\mathcal{T}_{c\ell} : w \mapsto z$

$$\mathcal{T}_{c\ell} := \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}\big(I - \mathcal{P}_{22}\mathcal{K}\big)^{-1}\mathcal{P}_{21}$$

- Find a stabilizing $\mathcal{K}$ that minimizes $\|\mathcal{T}_{c\ell}\|$.

Decentralized control



- $\mathcal{K}$ has sparsity pattern $\mathcal{S}$. e.g.

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_{11} & 0 \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

- **Assumption:** $\mathcal{P}_{22}$ also has sparsity pattern $\mathcal{S}$.

Main contribution

A decentralized version of the state-space $\mathcal{H}_\infty$ result [Doyle, Glover, Khargonekar, Francis '89] (DGKF)

- existence condition (iff)
- structural result
- computational result

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ cost

$\mathcal{H}_2$ cost: $$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}\big(G(j\omega)^* G(j\omega)\big)\,\mathrm{d}\omega$$

$\mathcal{H}_\infty$ cost: $$\|G\|_\infty = \operatorname*{ess\,sup}_{\omega}\,\bar{\sigma}\big(G(j\omega)\big)$$

$\mathcal{H}_2$ cost has advantages...

- is separable: $\left\| \begin{bmatrix} G_1 & G_2 \end{bmatrix} \right\|_2^2 = \|G_1\|_2^2 + \|G_2\|_2^2$
- is vectorizable: $\|A + BXC\|_2^2 = \|a + Mx\|_2^2$
- is smooth

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ cost

$\mathcal{H}_2$ **cost:** $$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}\big(G(j\omega)^* G(j\omega)\big)\,\mathrm{d}\omega$$

$\mathcal{H}_\infty$ **cost:** $$\|G\|_\infty = \sup_\omega \bar{\sigma}\big(G(j\omega)\big)$$

Minimum-entropy surrogate

$$\operatorname{ent}(G,\gamma) = \frac{-\gamma^2}{2\pi}\int_{-\infty}^{\infty} \log\left|\det\Big(I - \gamma^{-2} G(j\omega)^* G(j\omega)\Big)\right| \mathrm{d}\omega$$

key properties:

- entropy is smooth, strictly convex.
- $\operatorname{ent}(G,\gamma) < \infty \quad \Longleftrightarrow \quad \|G\|_\infty < \gamma$
- $\operatorname{ent}(G,\infty) = \|G\|_2^2$

Standard approach

Instead of minimizing $\|\mathcal{T}_{c\ell}\|_\infty$, use the fact that

$$\operatorname{ent}(\mathcal{T}_{c\ell}, \gamma) < \infty \iff \|\mathcal{T}_{c\ell}\|_\infty < \gamma$$

1. choose γ.

2. Find stabilizing $\mathcal{K}$ that minimizes $\operatorname{ent}(\mathcal{T}_{c\ell}, \gamma)$

3. $\begin{cases} \text{Finite minimum:} & \|\mathcal{T}_{c\ell}\|_\infty < \gamma. \text{ Decrease } \gamma \text{ and repeat.} \\ \text{Otherwise:} & \|\mathcal{T}_{c\ell}\|_\infty > \gamma. \text{ Increase } \gamma \text{ and repeat.} \end{cases}$

DGKF result

DGKF result

Preliminaries

State-space realization for the plant:

$$\begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

Assumptions:

- (A, B_2) stabilizable, (C_2, A) detectable
- D_{12} has full column rank, D_{21} has full row rank
- additional technical assumptions

DGKF result

There exists a stabilizing controller with $\|\mathcal{T}_{c\ell}\|_\infty < \gamma$ if and only if

(i) $X \succeq 0$ satisfies a Riccati equation.

(ii) $Y \succeq 0$ satisfies a Riccati equation.

(iii) $\rho(XY) < \gamma^2$

- One possible realization:

$$\mathcal{K} = \left[\begin{array}{c|c} A + B_2 K + ZLC_2 - \gamma^{-2} B_1 B_1^\mathsf{T} X & -ZL \\ \hline K & 0 \end{array}\right]$$

with $K := -B_2^\mathsf{T} X$, $L := -Y C_2^\mathsf{T}$, and $Z := (I - \gamma^{-2} Y X)^{-1}$

- $\mathcal{K}$ also minimizes $\mathrm{ent}(\mathcal{T}_{c\ell}, \gamma)$. [Glover, Mustafa '89]

Main result

Main result

Preliminaries

State-space realization for the plant:

$$\begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \qquad A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \text{ etc.}$$

Assumptions:

- (A, B_2) stabilizable, (C_2, A) detectable
- (A_{22}, B_{22}) stabilizable, (C_{11}, A_{11}) detectable
- D_{12} has full column rank, D_{21} has full row rank
- additional technical assumptions

Main result

There exists a triangular stabilizing controller with $\|\mathcal{T}_{c\ell}\|_\infty < \gamma$ iff

(i) there exists a stabilizing controller with $\|\mathcal{T}_{c\ell}\|_\infty < \gamma$

(ii) There exists $\hat{X}$ and $\hat{Y}$ such that

$$\begin{cases} \hat{X} - X \succeq 0 \text{ satisfies a Riccati equation that depends on } \hat{Y}. \\ \hat{Y} - Y \succeq 0 \text{ satisfies a Riccati equation that depends on } \hat{X}. \end{cases}$$

where both $\rho(X\hat{Y}) < \gamma^2$ and $\rho(\hat{X}Y) < \gamma^2$.

A realization for the controller that minimizes $\mathrm{ent}(\mathcal{T}_{c\ell}, \gamma)$ is

$$\left[\begin{array}{cc|c} A + B_2K + Z_L\hat{L}C_2 + \gamma^{-2}B_1B_1^\mathsf{T}X & 0 & -Z_L\hat{L} \\ B_2(K - \hat{K}Z_KZ^{-1}) & A + B_2\hat{K}Z_K + LC_2 + \gamma^{-2}YC_1^\mathsf{T}C_1 & -L \\ \hline K - \hat{K}Z_KZ^{-1} & \hat{K}Z_K & 0 \end{array}\right]$$

where $Z_K := (I - \gamma^{-2}Y\hat{X})^{-1}$ and $Z_L := (I - \gamma^{-2}\hat{Y}X)^{-1}$.

Comparison: $\mathcal{H}_\infty$

Centralized:

$$\mathcal{K} = \left[\begin{array}{c|c} A + B_2K + ZLC_2 - \gamma^{-2}B_1B_1^\mathsf{T}X & -ZL \\ \hline K & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} A + B_2KZ + LC_2 - \gamma^{-2}YC_1^\mathsf{T}C_1 & -L \\ \hline KZ & 0 \end{array}\right]$$

Decentralized:

$$\left[\begin{array}{cc|c} A + B_2K + Z_L\hat{L}C_2 + \gamma^{-2}B_1B_1^\mathsf{T}X & 0 & -Z_L\hat{L} \\ B_2(K - \hat{K}Z_KZ^{-1}) & A + B_2\hat{K}Z_K + LC_2 + \gamma^{-2}YC_1^\mathsf{T}C_1 & -L \\ \hline K - \hat{K}Z_KZ^{-1} & \hat{K}Z_K & 0 \end{array}\right]$$

$\mathcal{H}_2$ is limit $\gamma \to \infty$

Comparison: $\mathcal{H}_2$

Centralized:

$$\mathcal{K} = \left[\begin{array}{c|c} A + B_2K + LC_2 & -L \\ \hline K & 0 \end{array}\right]$$

Decentralized:

$$\left[\begin{array}{cc|c} A + B_2K + \hat{L}C_2 & 0 & -\hat{L} \\ B_2(K - \hat{K}) & A + B_2\hat{K} + LC_2 & -L \\ \hline K - \hat{K} & \hat{K} & 0 \end{array}\right]$$

Recovers the $\mathcal{H}_2$ result [Lessard, Lall '11] and more recently [Tanaka, Parrilo '14]

Computation

There exists $\hat{X}$ and $\hat{Y}$ such that

$$\begin{cases} \hat{X} - X \succeq 0 \text{ satisfies a Riccati equation that depends on } \hat{Y}. \\ \hat{Y} - Y \succeq 0 \text{ satisfies a Riccati equation that depends on } \hat{X}. \end{cases}$$

where both $\rho(X\hat{Y}) < \gamma^2$ and $\rho(\hat{X}Y) < \gamma^2$.

For fixed γ

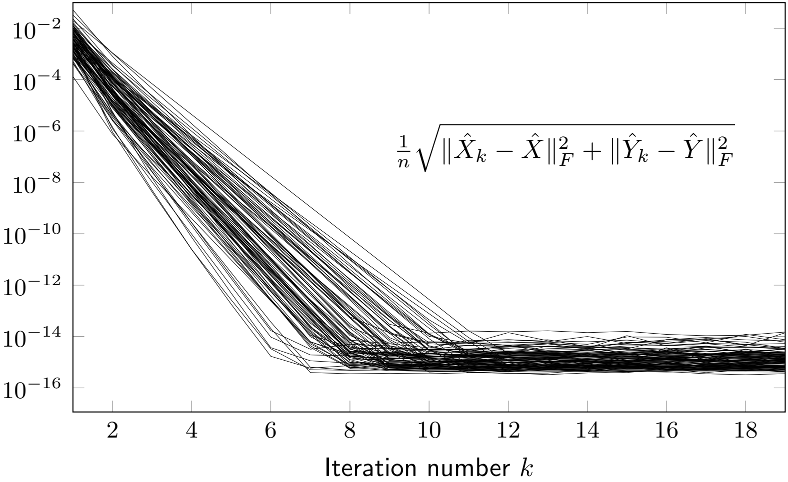
1. Guess $\hat{Y}_0$
2. $\hat{X}_{k+1} = \phi_1(\hat{Y}_k)$
3. $\hat{Y}_{k+1} = \phi_2(\hat{X}_{k+1})$
4. repeat.

To minimize γ

1. Find $(\hat{X}^{(0)}, \hat{Y}^{(0)})$ for $\gamma = \infty$
2. Choose $\gamma^{(0)}$ sufficiently large
3. Initialize with $(\hat{X}^{(k)}, \hat{Y}^{(k)}, \gamma^{(k)})$ and solve for $(\hat{X}^{(k+1)}, \hat{Y}^{(k+1)})$.
4. Choose $\gamma^{(k+1)} < \gamma^{(k)}$ and repeat.

Numerical evidence

100 random systems, $n = 20$ states, $\gamma \approx 2\gamma_{\mathsf{opt}}$



Proof outline

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1. Convert to *convex* model-matching problem [QI]

$$\underset{\substack{\mathcal{K}\text{ stabilizing}\\ \mathcal{K}\in\mathcal{S}}}{\text{minimize}}\ \operatorname{ent}\left(\mathcal{P}_{11}+\mathcal{P}_{12}\mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21},\,\gamma\right)$$

becomes: $$\underset{\substack{\mathcal{Q}\text{ stable}\\ \mathcal{Q}\in\mathcal{S}}}{\text{minimize}}\ \operatorname{ent}\left(\mathcal{T}_1+\mathcal{T}_2\mathcal{Q}\mathcal{T}_3,\,\gamma\right)$$

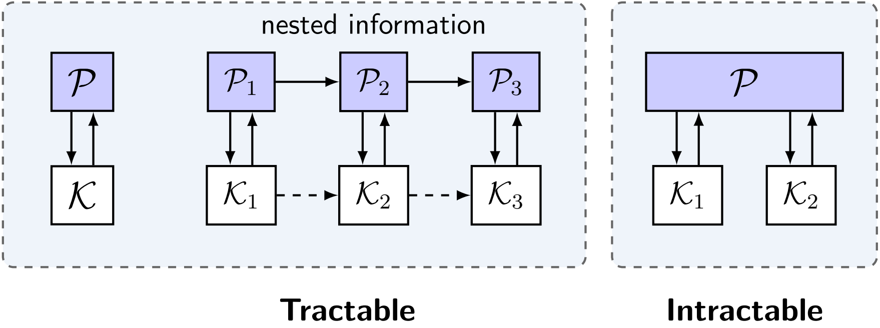
2. Use *person-by-person* approach [smoothness]

$$\underset{Q_{ij}\text{ stable}}{\text{minimize}}\ \operatorname{ent}\left(\mathcal{T}_1+\mathcal{T}_2\begin{bmatrix}\mathcal{Q}_{11} & 0\\ \mathcal{Q}_{21} & \mathcal{Q}_{22}\end{bmatrix}\mathcal{T}_3,\,\gamma\right)$$

- fix $\mathcal{Q}_{11}$ and solve for $\mathcal{Q}_{21}$, $\mathcal{Q}_{22}$. **(centralized!)**
- fix $\mathcal{Q}_{22}$ and solve for $\mathcal{Q}_{11}$, $\mathcal{Q}_{21}$. **(centralized!)**

3. Impose matching conditions (and lots of algebra)

Summary



Decentralized version of DGKF result

- Existence condition (iff)
- Explicit state-space formula
- Recover $\mathcal{H}_2$ result in limiting case
- Computational approach

Thank you!

Paper and additional information available at:
www.laurentlessard.com