State-space solution to a minimum-entropy $\mathcal{H}_\infty$-optimal control problem with a nested information constraint

Laurent Lessard[1]

IEEE Conference on Decision and Control, pp. 4026–4031, 2014

Abstract

State-space formulas are derived for the minimum-entropy $\mathcal{H}_\infty$ controller when the plant and controller are constrained to be block-lower-triangular. Such a controller exists if and only if: the corresponding unstructured problem has a solution, a certain pair of coupled algebraic Riccati equations admits a mutually stabilizing fixed point, and a pair of spectral radius conditions is met. The controller's observer-based structure is also discussed, and a simple numerical approach for solving the coupled Riccati equations is presented.

1 Introduction

Entropy minimization may be thought of as link between the popular $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance measures. Given a stable matrix transfer function (MTF) $\mathcal{F}(s)$, the entropy with tolerance $\gamma > 0$ is defined as

$$\mathcal{I}_\gamma(\mathcal{F}) := -\frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty} \log\left|\det\left(I - \gamma^{-2}\mathcal{F}(j\omega)^*\mathcal{F}(j\omega)\right)\right| d\omega \tag{1}$$

A key property of entropy is that $\mathcal{I}_\gamma(\mathcal{F})$ is finite if and only if $\|\mathcal{F}\|_\infty < \gamma$. So entropy may be used as a surrogate for the $\mathcal{H}_\infty$ norm when searching for suboptimal controllers with a prescribed performance γ. Entropy is also related to the $\mathcal{H}_2$ performance measure in the limit $\lim_{\gamma\to\infty} \mathcal{I}_\gamma(\mathcal{F}) = \|\mathcal{F}\|_2^2$.

The focus of this paper is a structurally constrained version of the standard $\mathcal{H}_\infty$ control problem whereby a given sparsity pattern is imposed on the controller. Such constraints arise in decentralized control; rows of the controller MTF $\mathcal{K}$ may be thought of as separate controllers and the constraint $\mathcal{K}_{ij} = 0$ means that controller i does not measure measurement j.

The plant and controller are continuous linear time-invariant systems described by the equations

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad \text{and} \qquad u = \mathcal{K}y$$

where z and w are the regulated output and exogenous input, and y and u are the measurement and controlled input, respectively. $\mathcal{P}_{22}$ and $\mathcal{K}$ are assumed to each have a 2×2 block-lower-triangular sparsity structure with conforming dimensions. That is:

$$\mathcal{P}_{22} := \begin{bmatrix} \mathcal{G}_{11} & 0 \\ \mathcal{G}_{21} & \mathcal{G}_{22} \end{bmatrix} \quad \text{and} \quad \mathcal{K} := \begin{bmatrix} \mathcal{K}_{11} & 0 \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix} \tag{2}$$

where $\mathcal{G}_{ij}$ is a strictly proper rational $k_i \times m_j$ MTF and $\mathcal{K}_{ij}$ is a proper rational $m_i \times k_j$ MTF. The closed-loop map $w \to z$ found by eliminating $\mathcal{K}$ is given by

$$\mathcal{T}_{c\ell} := \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} \tag{3}$$

The main problem statement is given below.

Min-entropy $\mathcal{H}_\infty$ two-player problem (METP) Given a plant $\mathcal{P}$ defined as above, find a controller $\mathcal{K}$ satisfying the following four requirements

(R1) $\mathcal{K}$ stabilizes $\mathcal{P}$.

(R2) The closed-loop map satisfies $\|\mathcal{T}_{c\ell}\|_\infty < \gamma$.

(R3) The entropy $\mathcal{I}_\gamma(\mathcal{T}_{c\ell})$ is minimized.

(R4) $\mathcal{K}$ has the triangular structure (2).

Requirements **(R1)**–**(R2)** describe the standard $\mathcal{H}_\infty$ control problem. Existing approaches include the seminal work by Doyle, Glover, Kargonekar, and Francis (DGKF) [1], and the linear matrix inequality (LMI) approach by Gahinet and Apkarian [4]. It was shown that the DGKF controller also minimizes entropy [5], as in **(R3)**. The risk-sensitive approach of Whittle [14] is also closely related to the entropy formulation. In the limit $\gamma \to \infty$, **(R1)**–**(R3)** reduce to the standard $\mathcal{H}_2$ optimal control problem. These various approaches and their interpretations are well covered in many modern texts on robust control. See for example [2, 15].

The structural constraint **(R4)** complicates the problem substantially. While the associated optimization may still be convexified [9, 10], it remains infinite-dimensional and there is no obvious way to find closed-form solutions. Nevertheless, **METP** was solved in the limiting $\mathcal{H}_2$ case by Lessard and Lall [7, 8]. In related work [12, 13], the $\mathcal{H}_2$ case was solved under the further assumption of noise-free state measurements (full-state feedback).

Unlike the limiting $\mathcal{H}_2$ case, **METP** for a general γ does not become appreciably simpler under full-state feedback assumptions. The only existing solution to the

[1]L. Lessard is with the Department of Mechanical Engineering at the University of California, Berkeley, CA 94720, USA.
lessard@berkeley.edu

general **METP** is by Scherer [11], and uses an LMI approach reminiscent of [4] together with a more general elimination lemma. The solution presented herein is completely different from [11] and may be thought of as a generalization of [1, 8] in that explicit formulas for the optimal controller are found. Both approaches are further compared in Section 3.

In the remainder of this section, a summary of notation and conventions is given. The main result and a short discussion are presented in Section 2. Implementation details are given in Section 4, and an outline of the proof is given in Section 5.

Common sets and operators. Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers respectively. Z^T and Z^* denote the transpose and conjugate transpose of Z, respectively. $\bar{\sigma}(Z)$ is the maximum singular value and $\rho(A)$ is the spectral radius. The imaginary axis is $j\mathbb{R}$. Let $\mathcal{L}_2^{m\times n}(j\mathbb{R})$ be the set of functions $\mathcal{F} : \mathbb{C} \to \mathbb{C}^{m\times n}$ such that the integral

$$\|\mathcal{F}\|_2^2 := \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}\bigl(\mathcal{F}(j\omega)^*\mathcal{F}(j\omega)\bigr)\mathrm{d}\omega$$

is bounded. The subspace $\mathcal{H}_2^{m\times n}(j\mathbb{R}) \subset \mathcal{L}_2^{m\times n}(j\mathbb{R})$ denotes the functions that are analytic in the open right-half plane. The shorthand $\mathcal{H}_2$ and $\mathcal{L}_2$ is used for brevity. The sets $\mathcal{H}_\infty$ and $\mathcal{L}_\infty$ are similarly defined, but with

$$\|\mathcal{F}\|_\infty := \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \bar{\sigma}\bigl(\mathcal{F}(j\omega)\bigr)$$

Let $\mathcal{R}_\mathrm{p}$ be the set of proper rational transfer matrices. Every $\mathcal{G} \in \mathcal{R}_\mathrm{p}$ has a state-space realization

$$\mathcal{G} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := D + C(sI - A)^{-1}B$$

If this realization is chosen to be stabilizable and detectable, then $\mathcal{G} \in \mathcal{H}_\infty$ if and only if A is Hurwitz, and $\mathcal{G} \in \mathcal{H}_2$ if and only if A is Hurwitz and $D = 0$. For a thorough introduction to these topics, see [15].

Structured matrices. The following notation is used to specify block-lower-triangular matrices.

$$\operatorname{lower}(S, m, n) := \left\{ \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix} \;\middle|\; X_{ij} \in S^{m_i\times n_j} \right\}$$

For such block matrices, also define

$$E_1 := \begin{bmatrix} I \\ 0 \end{bmatrix} \quad E^1 := \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad E_2 := \begin{bmatrix} 0 \\ I \end{bmatrix} \quad E^2 := \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

where the block dimensions are to be inferred by context. For example, if $A \in \operatorname{lower}(S, m, n)$, then $E_2^\mathsf{T} A E_1 = A_{21}$ and $AE^2A^\mathsf{T} = E_2 A_{22} A_{22}^\mathsf{T} E_2^\mathsf{T}$.

Hamiltonians. A Hamiltonian is a matrix of the form

$$H = \begin{bmatrix} A & R \\ -Q & -A^\mathsf{T} \end{bmatrix}$$

where A is square and R and Q are symmetric. If H has no eigenvalues on the imaginary axis, and satisfies the complementarity property [1, 15], then H is in the domain of the Riccati operator, written $H \in \operatorname{dom}(\mathrm{Ric})$. In this case, the associated algebraic Riccati equation (ARE) $A^\mathsf{T}X + XA + Q + XRX = 0$ has a unique solution such that $A + RX$ is Hurwitz. This stabilizing solution is denoted $X = \mathrm{Ric}(H)$, and it is always symmetric.

2 Main result

Suppose the plant $\mathcal{P} \in \mathcal{R}_\mathrm{p}$ has the state-space realization

$$\begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \tag{4}$$

in which the matrices A, B_2, C_2 have the structure

$$\begin{gathered} A \in \operatorname{lower}(\mathbb{R}, n, n),\, B_2 \in \operatorname{lower}(\mathbb{R}, n, m), \\ \text{and } C_2 \in \operatorname{lower}(\mathbb{R}, k, n). \end{gathered} \tag{5}$$

This ensures that $\mathcal{P}_{22}$ has the requisite block-lower-triangular structure (2). The converse is also true; whenever $\mathcal{P}_{22}$ satisfies (2), a realization satisfying (5) can be readily constructed [6, 9]. It is further assumed that A_{11} and A_{22} have non-empty dimensions. This avoids trivial special cases and allows for more streamlined results.

Finally, the same assumptions as in [1] are made on $\mathcal{P}_{22}$ in order to simplify the presentation.

(A1) (A, B_1) is stabilizable and (C_1, A) is detectable

(A2) (A, B_2) is stabilizable and (C_2, A) is detectable

(A3) $D_{12}^\mathsf{T}\begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$

(A4) $D_{21}\begin{bmatrix} B_1^\mathsf{T} & D_{21}^\mathsf{T} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$

As in [1], define the Hamiltonians

$$\begin{aligned} H_X &:= \begin{bmatrix} A & \gamma^{-2}B_1B_1^\mathsf{T} - B_2B_2^\mathsf{T} \\ -C_1^\mathsf{T}C_1 & -A^\mathsf{T} \end{bmatrix} \\ H_Y &:= \begin{bmatrix} A^\mathsf{T} & \gamma^{-2}C_1^\mathsf{T}C_1 - C_2^\mathsf{T}C_2 \\ -B_1B_1^\mathsf{T} & -A \end{bmatrix} \end{aligned} \tag{6}$$

For reference, recall the state-space solution of the classical $\mathcal{H}_\infty$ problem stated below as Theorem 1. Throughout this paper, we refer to the classical problem as *centralized* or *unstructured*, to distinguish it from **METP**.

Theorem 1 (DGKF [1])**.** *Suppose $\mathcal{P} \in \mathcal{R}_\mathrm{p}$ satisfies* (4) *as well as Assumptions* **(A1)**–**(A4)**. *There exists a controller that satisfies* **(R1)**–**(R3)** *if and only if*

(B1) $H_X \in \operatorname{dom}(\mathrm{Ric})$, *and* $X := \mathrm{Ric}(H_X) \geq 0$

(B2) $H_Y \in \operatorname{dom}(\mathrm{Ric})$, *and* $Y := \mathrm{Ric}(H_Y) \geq 0$

(B3) $\rho(XY) < \gamma^2$

When these conditions hold, one such controller is

$$\mathcal{K}_{\mathrm{cen}} = \left[\begin{array}{c|c} \hat{A} & -ZL \\ \hline K & 0 \end{array}\right] \tag{7}$$

where the following definitions were used.

$$\begin{aligned} \hat{A} &:= A + B_2K + ZLC_2 + \gamma^{-2}B_1B_1^\mathsf{T}X \\ Z &:= (I - \gamma^{-2}YX)^{-1} \\ K &:= -B_2^\mathsf{T}X \quad \textit{and} \quad L := -YC_2^\mathsf{T} \end{aligned} \tag{8}$$

The solution to the structured $\mathcal{H}_\infty$ problem involves a new pair of Hamiltonians. For any $\hat{Y} \ge 0$ such that $\rho(X\hat{Y}) < \gamma^2$, define

$$J_X(\hat{Y}) := \begin{bmatrix} A_X & R_X \\ -K^\mathsf{T}E^1K & -A_X^\mathsf{T} \end{bmatrix} \tag{9}$$

where the following definitions were used.

$$\begin{aligned} A_X &:= A + B_2E^2K + Z_L\hat{L}C_2 + \gamma^{-2}B_1B_1^\mathsf{T}X \\ R_X &:= \gamma^{-2}(B_1B_1^\mathsf{T} + Z_L\hat{L}\hat{L}^\mathsf{T}Z_L^\mathsf{T}) - B_2E^2B_2^\mathsf{T} \\ \hat{L} &:= -\hat{Y}C_2^\mathsf{T}E^1 \quad \text{and} \quad Z_L := (I - \gamma^{-2}\hat{Y}X)^{-1} \end{aligned} \tag{10}$$

Note that Z_L is invertible because $\rho(X\hat{Y}) < \gamma^2$. Similarly, for any $\hat{X} \ge 0$ such that $\rho(\hat{X}Y) < \gamma^2$, define

$$J_Y(\hat{X}) := \begin{bmatrix} A_Y^\mathsf{T} & R_Y \\ -LE^2L^\mathsf{T} & -A_Y \end{bmatrix} \tag{11}$$

where the following definitions were used.

$$\begin{aligned} A_Y &:= A + B_2\hat{K}Z_K + LE^1C_2 + \gamma^{-2}YC_1^\mathsf{T}C_1 \\ R_Y &:= \gamma^{-2}(C_1^\mathsf{T}C_1 + Z_K^\mathsf{T}\hat{K}^\mathsf{T}\hat{K}Z_K) - C_2^\mathsf{T}E^1C_2 \\ \hat{K} &:= -E^2B_2^\mathsf{T}\hat{X} \quad \text{and} \quad Z_K := (I - \gamma^{-2}Y\hat{X})^{-1} \end{aligned} \tag{12}$$

Note that the Hamiltonians J_X and J_Y are of the standard $\mathcal{H}_\infty$ type just like H_X and H_Y. That is, the constant term is positive semidefinite, while the quadratic term may be indefinite. The main result is given below.

Theorem 2. *Suppose $\mathcal{P} \in \mathcal{R}_\mathrm{p}$ satisfies* (4)–(5) *as well as Assumptions* **(A1)**–**(A4)**. *There exists a controller that solves* **METP** *if and only if*

(C1) *Conditions* **(B1)**–**(B3)** *hold. In other words, the unstructured version of the problem has a solution.*

(C2) *There exists $\hat{X}$ and $\hat{Y}$ such that*

$$\begin{cases} J_X(\hat{Y}) \in \mathrm{dom}(\mathrm{Ric}), \; \textit{and} \; \hat{X} - X = \mathrm{Ric}(J_X(\hat{Y})) \ge 0 \\ J_Y(\hat{X}) \in \mathrm{dom}(\mathrm{Ric}), \; \textit{and} \; \hat{Y} - Y = \mathrm{Ric}(J_Y(\hat{X})) \ge 0 \end{cases}$$

where both $\rho(X\hat{Y}) < \gamma^2$ and $\rho(\hat{X}Y) < \gamma^2$.

When these conditions hold, one such controller is

$$\mathcal{K}_{\mathrm{me}} := \left[\begin{array}{cc|c} \hat{A}_1 & 0 & -Z_L\hat{L} \\ B_2(K - \hat{K}Z_KZ^{-1}) & \hat{A}_2 & -L \\ \hline K - \hat{K}Z_KZ^{-1} & \hat{K}Z_K & 0 \end{array}\right] \tag{13}$$

where

$$\begin{aligned} \hat{A}_1 &:= A + B_2K + Z_L\hat{L}C_2 + \gamma^{-2}B_1B_1^\mathsf{T}X \\ \hat{A}_2 &:= A + B_2\hat{K}Z_K + LC_2 + \gamma^{-2}YC_1^\mathsf{T}C_1 \end{aligned}$$

and K, $\hat{K}$, L, $\hat{L}$, Z, Z_K, Z_L are defined in (6)–(12).

An outline of the proof is provided in Section 5. An immediate concern with Theorem 2 is that there is no obvious way to verify condition **(C2)**, as it requires solving two intricately coupled AREs. This point is discussed extensively in Section 4, where an efficient numerical method is proposed that can be used to find a fixed point when it exists. In the remainder of this section, some salient features of the optimal controller are discussed.

Coordinates. Let ξ be the state used in the realization of the optimal unstructured controller (7) in Theorem 1. If the state $\zeta := Z^{-1}\xi$ is used instead, the following dual realization is obtained.

$$\mathcal{K}_{\mathrm{cen}} = \left[\begin{array}{c|c} A + B_2KZ + LC_2 + \gamma^{-2}YC_1^\mathsf{T}C_1 & -L \\ \hline KZ & 0 \end{array}\right] \tag{14}$$

Likewise, there are many possible coordinate choices for representing the optimal structured controller (13).

In the closely related treatment of risk-sensitive optimal control by Whittle [14], the state ξ has the interpretation of extremizing *total stress*, while ζ extremizes *past stress*. In Theorem 2, the controller is expressed in *mixed* coordinates (ξ^1, ζ^2). That is, a ξ-like coordinate for the first state and a ζ-like coordinate for the second. This choice was made because it yields the simplest-looking formulae for $\mathcal{K}_{\mathrm{me}}$. Coordinate choice is further discussed in the proof outline in Section 5.

Structure. The controller $\mathcal{K}_{\mathrm{me}}$ may also be written in the standard observer form. The first state equation is

$$\dot{\xi}^1 = A\xi^1 + B_1\hat{w}^1 + B_2\hat{u}^1 - Z_L\hat{L}(y - C_2\xi^1) \tag{15}$$

where $\hat{w}^1 := \gamma^{-2}B_1^\mathsf{T}X$ and $\hat{u}^1 := K\xi^1$ are precisely the *worst-case noise* and *optimal input* respectively in the classical case [1]. In fact, (15) is identical to the classical centralized estimator except Y has been replaced by $\hat{Y}$. The second state equation is

$$\dot{\zeta}^2 = A\zeta^2 + B_2u + \gamma^{-2}YC_1^\mathsf{T}C_1 - L(y - C_2\zeta^2) \tag{16}$$

which is the estimator from $\mathcal{K}_{\mathrm{cen}}$ expressed in the ζ-coordinates, but with the optimal two-player u from Theorem 2 rather than the centralized $u = K\xi$ from Theorem 1. These structural properties suggest that $\mathcal{K}_{\mathrm{me}}$ exhibits a separation structure similar to the one described in [1], but further work is needed to state it precisely.

Limiting behavior. As mentioned in Section 1, entropy tends to the squared $\mathcal{H}_2$-norm in the limit $\gamma \to \infty$. When this limit is considered for the DGKF controller, then $Z = I$, $\zeta = \xi$ and the two realizations of $\mathcal{K}_{\mathrm{cen}}$ from (7) and (14) coincide.

Now consider the limit $\gamma \to \infty$ for $\mathcal{K}_{\mathrm{me}}$ in (13). Then $Z = Z_L = Z_K = I$ and the $\mathcal{H}_2$-optimal structured controller from [7] is recovered:

$$\mathcal{K}_{\mathrm{rn}} := \left[\begin{array}{cc|c} A + B_2K + \hat{L}C_2 & 0 & -\hat{L} \\ B_2(K - \hat{K}) & A + B_2\hat{K} + LC_2 & -L \\ \hline K - \hat{K} & \hat{K} & 0 \end{array}\right]$$

It can also be shown that in the limit $\gamma \to \infty$, the complicated condition **(C2)** simplifies to the simple linearly coupled equations given in [7].

3 Comparison with LMI method

The only existing solution to **METP** is the work by Scherer [11], which finds an LMI characterization of the γ-suboptimal controllers. Lower-triangular structures with more than two players are also considered in [11]. As in the classical case, there are many benefits to using an LMI approach; for example it allows a seamless treatment of singular problems [4]. The DGKF solution [1] makes more assumptions and is therefore more limited in its applicability. However, the DGKF solution provides observer-based formulas that give a clear and powerful interpretation of the controller's role and structure.

With regards to computational complexity, the DGKF solution is more efficient than the LMI approach. If $A \in \mathbb{R}^{n\times n}$, then the complexity of testing **(B1)**–**(B3)** is dominated by solving two AREs and finding one spectral radius. These are essentially eigenvalue problems and can be solved in $\mathcal{O}(n^3)$. The LMI formulation results in a semidefinite program (SDP) with two $n \times n$ decision variables. It therefore has a complexity of $\mathcal{O}(n^6)$ when using a conventional interior-point method.

For **METP**, the LMI test by Scherer [11] has more variables than the centralized LMI solution, but nevertheless has complexity $\mathcal{O}(n^6)$. Verifying the conditions in Theorem 2 is dominated by the task of finding $\hat{X}$ and $\hat{Y}$ that satisfy **(C2)**. To this end, a simple and efficient algorithm is given in Section 4 that roughly amounts to iteratively solving each ARE until convergence is achieved. Each step has complexity $\mathcal{O}(n^3)$ and it is verified empirically that convergence to machine precision takes fewer than 15 iterations and is independent of n.

Despite the proposed iterative ARE method being more computationally efficient than the LMI approach, both involve a necessary and sufficient condition for **METP**. One therefore expects there to exist a transformation of the coupled AREs of Theorem 2 into the LMI condition of [11] and vice versa. Such a construction for the centralized case is detailed in [3], but is the subject of future work for the unstructured case.

4 Iterative solution

As mentioned in Section 2, it is not clear how one would verify **(C2)** in Theorem 2. In this section, preliminary results are presented that suggest that a simple iterative scheme may be used to efficiently verify **(C2)**.

Iterative scheme (ITS). Given some $\gamma > 0$ and a starting guess $\hat{Y}_0$, solve the following AREs iteratively

$$\begin{aligned}\hat{X}_{k+1} &= X + \mathrm{Ric}(J_X(\hat{Y}_k))\\ \hat{Y}_{k+1} &= Y + \mathrm{Ric}(J_Y(\hat{X}_{k+1}))\end{aligned} \qquad \text{for } k = 0, 1, \dots \qquad (17)$$

and stop when $\hat{X}_k$ and $\hat{Y}_k$ have converged to some $\hat{X}$ and $\hat{Y}$ respectively. Then check to see if $\hat{X} \geq X$, $\hat{Y} \geq Y$, $\rho(X\hat{Y}) < \gamma^2$, and $\rho(\hat{X}Y) < \gamma^2$. If so, then **(C2)** is verified. If these conditions are not met, or if $J_X \notin \mathrm{dom}(\mathrm{Ric})$ or $J_Y \notin \mathrm{dom}(\mathrm{Ric})$ for any of the iterates, then the test is inconclusive.

Rapid convergence. The main issue with **ITS** is choosing a suitable initial point. In other words, finding $\hat{Y}_0$ such that $J_X(\hat{Y}_0) \in \mathrm{dom}(\mathrm{Ric})$. This task becomes increasingly difficult as γ approaches γ_{opt}, the infimum over all γ that solves **METP**. When $\gamma \to \infty$, **(C2)** is satisfied by the $\mathcal{H}_2$ values of $\hat{X}$ and $\hat{Y}$. Therefore, these limiting values of $\hat{X}$ and $\hat{Y}$, which are easily computed as in [7, 8], are good initializations for **ITS** when γ is sufficiently large.

To investigate this initialization, random structured systems with n states, and $\frac{n}{2}$ inputs and outputs were generated. The MATLAB function `rss` was used to generate (A_{11}, B_{11}, C_{11}) and (A_{22}, B_{22}, C_{22}), while `randn` was used to generate A_{21}, B_{21}, C_{21}, B_1, C_1. Matrices D_{12} and D_{21} were chosen to satisfy **(A3)**–**(A4)**. Finally, B_1 and C_1 were each scaled by $1/\sqrt{n}$. The result was a family of systems for which the infimal centralized γ_{cen} is approximately 3 for all n. For each test, γ_{opt} was approximated using the LMI method [11] and then **ITS** was performed for $\gamma = 2\gamma_{\mathrm{opt}}$ using the initialization described above. 100 tests were performed for each $n \in \{4, 8, 12, 16, 20\}$. Valid $\hat{X}$ and $\hat{Y}$ satisfying **(C2)** were successfully obtained in every case. In Figure 1, the convergence error

$$e_k := \frac{1}{n}\sqrt{\|\hat{X}_k - \hat{X}\|_F^2 + \|\hat{Y}_k - \hat{Y}\|_F^2}$$

is plotted a function of the iteration k for the case $n = 20$.

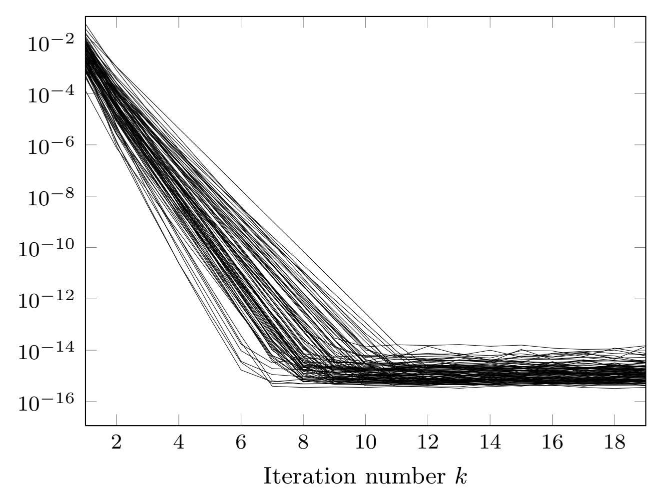

Figure 1: Convergence error e_k for the **ITS** algorithm. 100 random systems with $n = 20$ states at $\gamma \approx 2\gamma_{\mathrm{opt}}$.

No appreciable difference in convergence rate was observed as the state dimension n was increased; convergence was achieved in fewer than 15 iterations every time.

Note that rss only produces stable systems. If unstable modes are included in A, then even the optimal *centralized* γ_{cen} becomes very volatile and can sometimes exceed 10^4 depending on the number of unstable modes. The convergence of **ITS** is still linear in the presence of instability, but the rate is typically worse and more variable than the stable case, often taking 30–50 iterations to reach convergence. The slowest cases tested took up to 200 iterations. Nevertheless, the performance of **ITS** still appears to be independent of n as in the stable case.

Warm-start technique. When γ is too close to γ_{opt}, initializing $\hat{Y}_0$ with the limiting value of $\hat{Y}$ is sometimes ineffective. One possible solution is to iteratively decrease γ and set $\hat{Y}_0$ to be the converged $\hat{Y}$ from the previous γ iteration. Preliminary simulations indicate that this warm-started approach works very well, and γ_{opt} is achieved as long as γ is not decreased too rapidly.

5 Outline of the proof

The proof of Theorem 2 is algebraically involved, but conceptually simple. Due to space constraints, an outline of the proof is given that highlights the key enabling insights. The conditions **(C1)**–**(C2)** are necessary and sufficient for there to exist a solution to **METP**; each direction is addressed separately.

Proof of sufficiency. Suppose that **(C1)**–**(C2)** hold. The aim is to verify **(R1)**–**(R4)** separately.

It is immediate that **(R4)** is satisfied because each of the state-space matrices in (13) is block-lower-triangular. This follows from the sparsity of $\hat{L}$ and $\hat{K}$.

Requirements **(R1)**–**(R2)** are verified by appealing to the bounded real lemma and a well-known relationship between an ARE its associated algebraic Riccati inequality (ARI). We state these results as lemmas.

Lemma 3 (bounded real lemma). *Suppose $\mathcal{G}$ has realization $(A,B,C,0)$ and $\gamma > 0$ is given. The following statements are equivalent.*

(i) A is Hurwitz and $\|\mathcal{G}\|_\infty < \gamma$

(ii) There exists $X > 0$ such that $A^{\mathsf{T}}X + XA + \gamma^{-2}XBB^{\mathsf{T}}X + C^{\mathsf{T}}C < 0$

Lemma 4. *Suppose R and Q are symmetric and either $R \geq 0$ or $Q \geq 0$. Then the following are equivalent*

(i) There exists $X > 0$ satisfying the inequality $A^{\mathsf{T}}X + XA + XRX + Q < 0$.

(ii) There exists $X_0 \geq 0$ such that $A + RX_0$ is Hurwitz and $A^{\mathsf{T}}X_0 + X_0A + X_0RX_0 + Q = 0$

If the above conditions hold, then $0 \leq X_0 < X$.

Applying Lemma 3 to the closed-loop map $\mathcal{T}_{c\ell}$ (3) induced by the proposed $\mathcal{K}_{\text{me}}$, we find *(i)* is equivalent to **(R1)**–**(R2)**. Rather than solving the inequality in Lemma 3, a stabilizing solution to the associated ARE is constructed and Lemma 4 is applied.

There are many possible realizations for $\mathcal{T}_{c\ell}$ to choose from. If the plant $\mathcal{P}$ has state x, and $\mathcal{K}_{\text{me}}$ has states (ξ^1, ζ^2), then define the coordinate choices

$$m : \begin{bmatrix} \xi^1 \\ x - \xi^1 \\ x - \zeta^2 \end{bmatrix} \qquad x : \begin{bmatrix} x \\ x - \xi^1 \\ x - \xi^2 \end{bmatrix} \qquad y : \begin{bmatrix} \zeta^1 \\ \zeta^2 - \zeta^1 \\ x - \zeta^2 \end{bmatrix}$$

The new coordinates ξ^2 and ζ^1 are related to the states of $\mathcal{K}_{\text{me}}$ as follows.

$$\begin{bmatrix} \zeta^1 \\ \zeta^2 \end{bmatrix} = \begin{bmatrix} Z_L^{-1} & 0 \\ Z^{-1} - Z_K^{-1} & Z_K^{-1} \end{bmatrix} \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix}$$

This relationship is a generalization of the classical $\mathcal{H}_\infty$ coordinate transform $\zeta = Z^{-1}\xi$ mentioned in Section 2. The x and y coordinates were chosen because they yield simple solutions to the bounded real equation. The result is given by the following lemma.

Lemma 5. *Consider the setting of Theorem 2. Express the closed-loop map $\mathcal{T}_{c\ell}$ in the x and y coordinates, and associate the following labels to the corresponding state-space matrices.*

$$\mathcal{T}_{c\ell} = \left[\begin{array}{c|c} \bar{A}_x & \bar{B}_x \\ \hline \bar{C}_x & 0 \end{array}\right] = \left[\begin{array}{c|c} \bar{A}_y & \bar{B}_y \\ \hline \bar{C}_y & 0 \end{array}\right] \tag{18}$$

Define the associated Hamiltonians

$$\bar{H}_X := \begin{bmatrix} \bar{A}_x & \gamma^{-2}\bar{B}_x\bar{B}_x^{\mathsf{T}} \\ -\bar{C}_x^{\mathsf{T}}\bar{C}_x & -\bar{A}_x^{\mathsf{T}} \end{bmatrix}, \bar{H}_Y := \begin{bmatrix} \bar{A}_y^{\mathsf{T}} & \gamma^{-2}\bar{C}_y^{\mathsf{T}}\bar{C}_y \\ -\bar{B}_y\bar{B}_y^{\mathsf{T}} & -\bar{A}_y \end{bmatrix}$$

Then $\bar{H}_X \in \operatorname{dom}(\operatorname{Ric})$ and $\bar{H}_Y \in \operatorname{dom}(\operatorname{Ric})$. Moreover,

$$\operatorname{Ric}(\bar{H}_X) = \begin{bmatrix} X & 0 & 0 \\ 0 & \hat{X} - X & 0 \\ 0 & 0 & \Phi \end{bmatrix}, \operatorname{Ric}(\bar{H}_Y) = \begin{bmatrix} \Psi & 0 & 0 \\ 0 & \hat{Y} - Y & 0 \\ 0 & 0 & Y \end{bmatrix}$$

where X, Y, $\hat{X}$, $\hat{Y}$ are defined in **(B1)**–**(B2)** *and* **(C2)**, *and $\Phi \geq 0$ and $\Psi \geq 0$.*

The variables Φ and Ψ from Lemma 5 are also solutions to AREs, but the associated formulas are omitted because the values of Φ and Ψ are unimportant.

By Lemma 4, the solutions to the AREs given in Lemma 5 imply that the bounded real inequalities also have solutions, and so by Lemma 3, $\mathcal{T}_{c\ell}$ has a stable state-space realization and $\|\mathcal{T}_{c\ell}\|_\infty < \gamma$. In other words, requirements **(R1)**–**(R2)** have been verified.

Verifying **(R3)**, or that the proposed controller minimizes entropy, is accomplished by first deriving a necessary and sufficient condition for optimality and then checking that it is satisfied by the proposed $\mathcal{K}_{\text{me}}$.

Lemma 6. *Suppose the set of admissible closed-loop maps is parameterized by*

$$\mathcal{T}_{c\ell} \in \{\mathcal{T}_1 + \mathcal{T}_2 \mathcal{Q} \mathcal{T}_3 \mid \mathcal{Q} \in \operatorname{lower}(\mathcal{H}_2, m, k)\}$$

where the $\mathcal{T}_i$ are stable. If $\|\mathcal{T}_{c\ell}\|_\infty < \gamma$, then $\mathcal{T}_{c\ell}$ has minimum entropy if and only if

$$\mathcal{T}_2^* \mathcal{T}_{c\ell}(I - \gamma^{-2}\mathcal{T}_{c\ell}^* \mathcal{T}_{c\ell})^{-1} \mathcal{T}_3^* \in \begin{bmatrix} \mathcal{H}_2^\perp & \mathcal{L}_2 \\ \mathcal{H}_2^\perp & \mathcal{H}_2^\perp \end{bmatrix} \tag{19}$$

A similar approach was taken in [8] to prove optimality in the $\mathcal{H}_2$ case. Indeed, in the limit $\gamma \to \infty$, Equation (19) becomes the $\mathcal{H}_2$ optimality condition from [8].

It has already been verified that $\|\mathcal{T}_{c\ell}\|_\infty < \gamma$, so it remains to show that the optimality condition holds. The closed-loop map may be expressed in the affine form $\mathcal{T}_{c\ell} = \mathcal{T}_1 + \mathcal{T}_2 \mathcal{Q} \mathcal{T}_3$ using a modified Youla parameterization. The following result is from [8]. Similar parameterizations were also reported in [6] and [9].

Lemma 7. *Suppose $\mathcal{P}$ and its state-space realization satisfies* (4)–(5)*. There exists $\mathcal{K} \in \operatorname{lower}(\mathcal{R}_\mathrm{p}, m, k)$ that stabilizes $\mathcal{P}$ if and only if (C_{ii}, A_{ii}, B_{ii}) is stabilizable and detectable for $i = 1, 2$. In this case, let K_i and L_i be such that $A + B_{ii}K_i$ and $A + L_i C_{ii}$ are Hurwitz. Then define $K_d := \operatorname{diag}(K_1, K_2)$ and $L_d := \operatorname{diag}(L_1, L_2)$. the set of all stabilized closed-loop maps is parameterized by*

$$\mathcal{T}_{c\ell} \in \{\mathcal{T}_1 + \mathcal{T}_2 \mathcal{Q} \mathcal{T}_3 \mid \mathcal{Q} \in \operatorname{lower}(\mathcal{H}_2, m, k)\} \tag{20}$$

where the $\mathcal{T}_i$ matrices have the joint realization

$$\begin{bmatrix} \mathcal{T}_1 & \mathcal{T}_2 \\ \mathcal{T}_3 & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} A_{Kd} & -B_2 K_d & B_1 & B_2 \\ 0 & A_{Ld} & B_{Ld} & 0 \\ \hline C_{Kd} & -D_{12}K_d & 0 & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right] \tag{21}$$

and the following shorthand notation was used.

$$\begin{aligned} A_{Kd} &:= A + B_2 K_d & A_{Ld} &:= A + L_d C_2 \\ C_{Kd} &:= C_1 + D_{12} K_d & B_{Ld} &:= B_1 + L_d D_{21} \end{aligned} \tag{22}$$

There is also a one-to-one mapping between each stabilizing controller $\mathcal{K}$ and its associated $\mathcal{Q}$-parameter, but these details are left out of Lemma 7 to save some space. Using the parameterization of Lemma 7, the optimality condition (19) may be verified by direct substitution and appropriate state-space simplifications.

Proof of necessity. This part of the proof is similar to how necessity was proved in the $\mathcal{H}_2$ case [8]. Roughly, the $\mathcal{Q}_{11}$ part of the controller from the parameterization of Lemma 7 is held fixed and the problem of finding the minimum-entropy $\begin{bmatrix} \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix}$ is considered. This problem is unstructured so Theorem 1 may be applied. The result is that a pair of AREs must have positive-semidefinite solutions and a spectral radius condition must be met. After some algebraic manipulations, it is found that the AREs are those that correspond to the Hamiltonians J_X and H_Y and the spectral radius condition amounts to $\rho(\hat{X}Y) < \gamma^2$. Using a similar argument, holding $\mathcal{Q}_{22}$ fixed leads to the conditions on J_Y and H_X together with $\rho(X\hat{Y}) < \gamma^2$.

6 Acknowledgments

The author would like to thank Bo Bernhardsson, Sanjay Lall, Andrew Packard, and Peter Seiler for helpful discussions. This work was supported in part by NASA under Grant No. NRA NNX12AM55A.

References

[1] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis. State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems. *IEEE Transactions on Automatic Control*, 34(8):831–847, 1989.

[2] G. E. Dullerud and F. Paganini. *A Course in Robust Control Theory: A Convex Approach*. Springer Verlag, 2000.

[3] P. Gahinet. Explicit controller formulas for LMI-based $\mathcal{H}_\infty$ synthesis. *Automatica*, 32(7):1007–1014, 1996.

[4] P. Gahinet and P. Apkarian. A linear matrix inequality approach to $\mathcal{H}_\infty$ control. *International Journal of Robust and Nonlinear Control*, 4(4):421–448, 1994.

[5] K. Glover and D. Mustafa. Derivation of the maximum entropy $\mathcal{H}_\infty$-controller and a state-space formula for its entropy. *International Journal of Control*, 50(3):899–916, 1989.

[6] L. Lessard, M. Kristalny, and A. Rantzer. On structured realizability and stabilizability of linear systems. In *American Control Conference*, pages 5784–5790, 2013.

[7] L. Lessard and S. Lall. A state-space solution to the two-player decentralized optimal control problem. In *Allerton Conference on Communication, Control, and Computing*, pages 1559–1564, 2011.

[8] L. Lessard and S. Lall. Optimal control of two-player systems with output feedback. *To Appear, IEEE Transactions on Automatic Control*, 2014. arXiv:1303.3644.

[9] X. Qi, M. V. Salapaka, P. G. Voulgaris, and M. Khammash. Structured optimal and robust control with multiple criteria: a convex solution. *IEEE Transactions on Automatic Control*, 49(10):1623–1640, 2004.

[10] M. Rotkowitz and S. Lall. A characterization of convex problems in decentralized control. *IEEE Transactions on Automatic Control*, 51(2):274–286, 2006.

[11] C. W. Scherer. Structured $\mathcal{H}_\infty$-optimal control for nested interconnections: A state-space solution. *Systems & Control Letters*, 62(12):1105–1113, 2013.

[12] P. Shah and P. Parrilo. $\mathcal{H}_2$-optimal decentralized control over posets: A state-space solution for state-feedback. *IEEE Transactions on Automatic Control*, 58(12):3084–3096, 2013.

[13] J. Swigart and S. Lall. An explicit state-space solution for a decentralized two-player optimal linear-quadratic regulator. In *American Control Conference*, pages 6385–6390, 2010.

[14] P. Whittle. Risk-sensitive linear/quadratic/Gaussian control. *Advances in Applied Probability*, 13(4):764–777, 1981.

[15] K. Zhou, J. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall, 1995.