Performance Certification of Interconnected Nonlinear Systems using ADMM

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Abstract

We present a compositional performance certification method for interconnected nonlinear systems, using dissipativity properties of the subsystems along with the interconnection structure. To select the most relevant dissipativity properties, we formulate a large-scale optimization problem, and solve it with the alternating direction method of multipliers (ADMM). The dissipativity properties are allowed to depend on an unknown equilibrium, enabling us to certify performance without explicit knowledge of the equilibrium for the interconnected system. The effectiveness of the algorithm is demonstrated on two examples, including a model of vehicle platoons.

1   Introduction

In this paper, we consider compositional analysis for performance certification of an interconnection of nonlinear subsystems as depicted in Figure 1. The G_i blocks are known subsystems mapping $u_i \mapsto y_i$, with dynamics described by nonlinear state equations. M is a static matrix characterizing the interconnection topology. The overarching goal of compositional analysis is to establish properties of the interconnected system using only properties of the subsystems along with information about the interaction of the subsystems (M). Henceforth, the term "local" is used to refer to properties or analysis of individual subsystems in isolation. Likewise, "global" refers to the entire interconnected system.
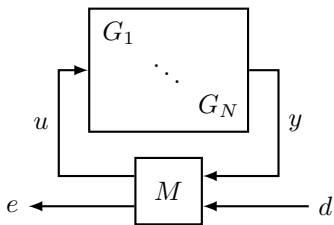

Figure 1: Interconnected system with exogenous input d and performance output e.

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In this paper, local and global properties are quantified in the framework of dissipative systems [13]; specifically the case with quadratic supply rates [14]. The choice of a supply rate dictates the specific property that is to be verified. For example, different supply rates can be used to verify L_2-gain properties, passivity, output-strict passivity, etc.

A conventional approach to compositional analysis, as presented for example in [1, 2, 10, 13] is to establish individual supply rates (and storage functions) for which each subsystem is dissipative. Then, a candidate storage function for the interconnected system is sought as a linear combination of the subsystem storage functions.

The method presented here is less conservative in that the local storage functions are treated as decision variables, and are optimized with regards to their particular suitability in certifying desired properties of the interconnected system. Thus, global properties are certified via local certificates that have been automatically generated, as opposed to having been preselected.

The idea of optimizing over the local supply rates (and storage functions) to certify stability of an interconnected system was introduced in [12], with the local supply rates constrained to be diagonally-scaled induced L_2-norms. This perspective coupled with dual decomposition, gave rise to a distributed optimization problem to certify the stability of the interconnection. More recent work in [9] generalized this in several ways: certifying dissipativity (rather than stability) of the interconnected system with respect to a quadratic supply rate; using arbitrary quadratic supply rates for the local subsystems; and employing ADMM [3] which exposes the distributed certification as a convergent negotiation between local computational agents for each subsystem and a master agent. The roles of the local and master agents are as follows:

- Each local agent receives a proposed supply rate from the master agent and solves an optimization problem to find a supply rate that certifies a local dissipativity property and is close to the proposed supply rate.

- The master agent, with knowledge of the interconnection topology M and the updated supply rates, solves an optimization problem to certify a global dissipativity property and proposes new supply rates to the local agents.

In this paper, we continue along the direction proposed in [9]. Specifically, we apply this method to nonlinear subsystems using sum-of-squares (SOS) optimization and use the concept of equilibrium-independent dissipativity (EID) for interconnections where the equilibrium point itself is unknown.

2   Preliminaries

Dissipativity theory [13].   Consider a time-invariant, continuous-time dynamical system described by

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)), \quad f(0,0) = 0 \\
y(t) &= h(x(t), u(t)), \quad h(0,0) = 0
\end{aligned}
\tag{1}
$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$. A *supply rate* is a function $w : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$. A system of the form (1) is *dissipative* with respect to a supply rate w if there exists a differentiable and nonnegative function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$
\nabla V(x)^\mathsf{T} f(x, u) - w(u, y) \leq 0
\tag{2}
$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y = h(x, u)$. Equation (2) is referred to as the *Dissipation Inequality Equation* (DIE) and V as a *storage function*.

Equilibrium independent dissipativity [7].   EID can be used to analyze systems where the equilibrium point depends nontrivially on the system input. Consider a system of the form

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)) \\
y(t) &= h(x(t), u(t))
\end{aligned}
\tag{3}
$$

such that for each equilibrium point $x^* \in \mathbb{R}^n$ there exists a unique $u^* \in \mathbb{R}^m$ such that $f(x^*, u^*) = 0$. The *equilibrium state-input map* is then defined as

$$
k_u(x) : \mathbb{R}^n \to \mathbb{R}^m \text{ such that } u^* = k_u(x^*)
$$

and the *equilibrium state-output map* is defined as

$$
k_y(x) : \mathbb{R}^n \to \mathbb{R}^p \text{ such that } y^* = k_y(x^*) = h(x^*, k_u(x^*))
$$

A system of the form (3) is *EID* with respect to the supply rate, w, if there exists a nonnegative storage function, $V : \mathbb{R}^{2n} \to \mathbb{R}_+$, such that $V(x^*, x^*) = 0$ and

$$
\nabla_x V(x, x^*)^\mathsf{T} f(x, u) - w(u - u^*, y - y^*) \leq 0
\tag{4}
$$

for all equilibrium points x^*, and for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ where $y = h(x, u)$, $y^* = k_y(x^*)$, and $u^* = k_u(x^*)$.

Note that the definition given here is slightly different than in [7] and follows the convention in [4] and [5]. We assume the existence of an *equilibrium state-input* map whereas in [7] it is assumed that an *equilibrium input-state* map exists. Our definition allows systems such as an integrator to be EID.

SOS programming.   Let $\mathbb{R}[x]$ be the set of all polynomials in x with real coefficients. $\Sigma[x] := \{ \pi \in \mathbb{R}[x] : \pi = \pi_1^2 + \cdots + \pi_n^2, \quad \pi_1, \ldots, \pi_n \in \mathbb{R}[x] \}$ is the subset of $\mathbb{R}[x]$ containing the SOS polynomials. A polynomial, $p(x)$, being a sum of squares polynomial is equivalent to the existence of a positive semidefinite matrix Q such that

$$
p(x) = m(x)^\mathsf{T} Q \, m(x)
\tag{5}
$$

for a properly chosen vector of monomials, $m(x)$. Therefore checking the nonnegativity of a polynomial can be relaxed to a SOS program and then solved as a semidefinite program (SDP).

Polynomial dynamics.   Suppose that f and h in (1) are polynomials. Then, certification of dissipativity of the system with respect to a polynomial supply rate w can be relaxed to a SOS feasibility program:

$$
\begin{aligned}
V(x) &\in \Sigma[x] \\
-\nabla V(x)^\mathsf{T} f(x, u) + w(u, y) &\in \Sigma[x, u]
\end{aligned}
\tag{6}
$$

Similarly, certification of EID for polynomial systems can be relaxed to a SOS feasibility program [7]:

$$
\begin{aligned}
V(x, x^*) &\in \Sigma[x, x^*] \\
r(x, x^*, u, u^*) &\in \mathbb{R}[x, u, x^*, u^*] \\
-\nabla_x V(x, x^*)^\mathsf{T} f(x, u) + w(u - u^*, y - y^*) & \\
+ r(x, x^*, u, u^*) f(x^*, u^*) &\in \Sigma[x, u, x^*, u^*]
\end{aligned}
\tag{7}
$$

Thus, for polynomial dynamics, a storage function that satisfies (2) or (4) can be found by solving (6) or (7), respectively.

Rational polynomial dynamics.   Suppose for each state variable x_i the associated system dynamics f_i are described by rational polynomials of the form

$$
f_i(x, u) = \frac{p_i(x, u)}{q_i(x, u)} \quad \text{for all } i = 1, \ldots, n
\tag{8}
$$

where $p_i \in \mathbb{R}[x, u]$ and $q_i - \epsilon \in \Sigma[x, u]$ for some positive ϵ. Then, certifying dissipativity of the system with respect to a polynomial supply rate w can be relaxed to a SOS feasibility program:

$$
\begin{aligned}
V(x) &\in \Sigma[x] \\
-\sum_{i=1}^{n} \nabla_{x_i} V(x) p_i(x, u) \prod_{j \neq i}^{n} q_j(x, u) & \\
+ \prod_{i=1}^{n} q_i(x, u) w(u, y) &\in \Sigma[x, u]
\end{aligned}
\tag{9}
$$

Thus, for rational polynomial dynamics, a storage function satisfying (9) can be found by solving (2). Similarly, to the polynomial case, certification of EID for rational polynomial systems can also be formulated as an SOS feasibility program.

3 Problem statement

Consider the interconnected system in Figure 1 which consists of N known subsystems, G_i, with a known, static interconnection $M \in \mathbb{R}^{m \times p}$. Therefore,

$$\begin{bmatrix} u \\ e \end{bmatrix} = M \begin{bmatrix} y \\ d \end{bmatrix} \tag{10}$$

and each G_i has dynamics of the form (1) and is characterized by a local state vector x_i. We assume that the interconnected system is well-posed, meaning that for any $d \in L_{2e}$, and any initial condition x_0 there exists unique $e, u, y \in L_{2e}$ that causally depend on d. We assume there exists an equilibrium point x^* for which there is a unique $u^* = k_u(x^*)$, $y^* = k_y(x^*)$, and $e^* = M_{21}y^*$.

We assume that the global and local supply rates are quadratic forms. The global supply rate is given by

$$\begin{bmatrix} d \\ e - e^* \end{bmatrix}^\mathsf{T} W \begin{bmatrix} d \\ e - e^* \end{bmatrix} \tag{11}$$

and the local supply rates are

$$\begin{bmatrix} u_i - u_i^* \\ y_i - y_i^* \end{bmatrix}^\mathsf{T} X_i \begin{bmatrix} u_i - u_i^* \\ y_i - y_i^* \end{bmatrix} \tag{12}$$

Here, W is give as part of the problem specification, and the X_i are real symmetric matrix decision variables. To certify the desired global dissipativity from local EID properties, we pose an optimization problem of the form

$$\begin{aligned} \underset{X_{1:N}}{\text{minimize}} \quad & 0 \\ \text{subject to} \quad & X_i \in \mathcal{L}_i \quad \text{for } i = 1, \dots, N \\ & (X_1, \dots, X_N) \in \mathcal{G} \end{aligned} \tag{13}$$

where each $\mathcal{L}_i$ constraint is *local* because it involves only the local supply rate X_i. The $\mathcal{G}$ constraint is *global* because it involves all the supply rates.

Before defining the $\mathcal{L}_i$ and $\mathcal{G}$ sets, we first introduce the following conformal block partitions

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \qquad X_i = \begin{bmatrix} X_i^{11} & X_i^{12} \\ X_i^{21} & X_i^{22} \end{bmatrix}$$

and the following block-diagonal matrices

$$X^{jk} = \begin{bmatrix} X_1^{jk} & & \\ & \ddots & \\ & & X_N^{jk} \end{bmatrix} \quad \text{for all } j, k \in \{1, 2\}$$

Recall that W is given while $X_1, \dots, X_N$ are to be found. The local and global sets are then defined as follows.

$$\mathcal{L}_i := \left\{ X_i \, \middle| \, \text{the } i\text{-th subsystem is EID w.r.t} \right.$$
$$\left. \text{the supply rate } \begin{bmatrix} u_i - u_i^* \\ y_i - y_i^* \end{bmatrix}^\mathsf{T} X_i \begin{bmatrix} u_i - u_i^* \\ y_i - y_i^* \end{bmatrix} \right\} \tag{14}$$

$$\mathcal{G} := \left\{ X_{1:N} \, \middle| \, \sum_{i=1}^{N} H_i X_i H_i^\mathsf{T} - H_0 W H_0^\mathsf{T} \preceq 0 \right\} \tag{15}$$

Here, the constant matrices $H_0, \dots, H_N$ are defined such that the following identity holds.

$$\sum_{i=1}^{N} H_i X_i H_i^\mathsf{T} - H_0 W H_0^\mathsf{T} =$$
$$\begin{bmatrix} M \\ I \end{bmatrix}^\mathsf{T} \begin{bmatrix} X^{11} & 0 & X^{12} & 0 \\ 0 & -W_{22} & 0 & -W_{21} \\ X^{21} & 0 & X^{22} & 0 \\ 0 & -W_{12} & 0 & -W_{11} \end{bmatrix} \begin{bmatrix} M \\ I \end{bmatrix} \tag{16}$$

In [9] it was shown that if each subsystem is dissipative with respect to a supply rate, X_i, and that $(X_1, \dots, X_N) \in \mathcal{G}$ then the interconnected system is dissipative with respect to the global supply rate W. This result extends directly to the EID case: if the subsystems are EID and the global constraint is satisfied then the interconnected system is dissipative.

The benefit of the formulation in (13) is that the local constraints only depend on the supply rate and storage function for the associated subsystem, so they can be checked separately and in parallel. The global constraint is an LMI that depends on all the supply rates, but does not depend on the storage functions.

Note that our approach seeks a global storage function that is additively separable in the subsystem states. Therefore, it is potentially conservative compared to searching for a storage function that is not separable.

4 ADMM

In [9] we demonstrated that ADMM [3] could be used to decompose and solve (13) reliably and efficiently. This method allows us to solve the individual subproblems, which require searching for a potentially high order storage function, in parallel.

ADMM can be used to solve problems of the form

$$\begin{aligned} \text{minimize} \quad & f(x) + g(z) \\ \text{subject to} \quad & Ax + Bz = c \end{aligned} \tag{17}$$

where x and z are vector decision variables. ADMM is guaranteed to converge if f, g are closed, proper, and convex, and the Lagrangian has a saddle point [3].

Our problem (13) may be put into the form of (17) by defining the following indicator functions:

$$\mathbb{I}_{\mathcal{L}_i}(X_i) := \begin{cases} 0 & X_i \in \mathcal{L}_i \\ \infty & \text{otherwise} \end{cases}$$

$$\mathbb{I}_{\mathcal{G}}(X_{1:N}) := \begin{cases} 0 & (X_1, \dots, X_N) \in \mathcal{G} \\ \infty & \text{otherwise} \end{cases}$$

Then (13) may be written equivalently as

$$\begin{aligned} \underset{X_{1:N}, Z_{1:N}}{\text{minimize}} \quad & \sum_{i=1}^{N} \mathbb{I}_{\mathcal{L}_i}(X_i) + \mathbb{I}_{\mathcal{G}}(Z_1, \dots, Z_N) \\ \text{subject to} \quad & X_i - Z_i = 0 \quad \text{for } i = 1, \dots, N \end{aligned}$$

so that it is in the canonical form (17). The first term in the objective and the constraints are separable for each subsystem, so the ADMM update takes on the following parallelized form [3].

1. X-updates: for each i, solve the local problem:

$$X_i^{k+1} = \underset{X \in \mathcal{L}_i}{\arg\min} \lVert X - Z_i^k + U_i^k \rVert_F^2$$

2. Z-update: if $(X_1^{k+1}, \ldots, X_N^{k+1}) \in G$, then we have found a solution to (13), so terminate the algorithm. Otherwise, solve the global problem:

$$Z_{1:N}^{k+1} = \underset{Z_{1:N} \in \mathcal{G}}{\arg\min} \left\lVert \sum_{i=1}^N (X_i^{k+1} - Z_i + U_i^k) \right\rVert_F^2$$

3. U-update: update U and return to step 1.

$$U_i^{k+1} = X_i^{k+1} - Z^{k+1} + U_i^k$$

5 Examples

The first example we present demonstrates the scalability and reliability of this approach for certifying performance of large nonlinear systems. The second example is based on the model of vehicle platoons used in [4] and [5].

A large-scale polynomial system The system consists of N, 2-state subsystems that share a common dynamical structure, modified from [8], but with different parameters defining each subsystem. An individual subsystem, H, has 2 states and is described by

$$H: \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-ax_2 - bx_1^3 + u}{1 + cx_2^2} \\ y &= x_2 \end{aligned} \quad (18)$$

where $a, b, c > 0$ are parameters of the subsystem. The positive-definite storage function $V(x) := \frac{ab}{2}x_1^4 + \frac{ac}{2}x_2^4 + ax_2^2$ and the supply rate $w(u,y) := u^2 - a^2 y^2$ certify that the L_2-gain of H is bounded by a^{-1}.

Clearly, *any* well-posed interconnection involving many instances of these systems, along with an interconnection matrix whose spectral norm is less than a results in a dynamical systems whose L_2 gain is less than 1. Armed with this insight, the following steps were used to construct examples of large-scale interconnected systems:

1. Choose $\{a_i, b_i, c_i\}_{i=1}^N$ uniformly distributed in $(1,2) \times (0,1) \times (0.5,2)$. These constitute the parameters of system H_i. Denote $\gamma := \max_i a_i^{-1}$.

2. Draw each entry of $S \in \mathbb{R}^{(N+d) \times (N+d)}$ from a standard normal distribution. Overlay a desired sparsity pattern by selectively zeroing out particular entries.

3. Compute $\beta := \inf_B \bar{\sigma}(BSB^{-1})$ where $B = \mathrm{diag}(b_1, \ldots, b_N, I_d)$, $b \in \mathbb{R}_{++}^N$. Redefine $S := \frac{0.99}{\gamma\beta}S$ to ensure that its spectral norm is less than γ.

4. Choose random nonzero, diagonal scalings $\Psi = \mathrm{diag}(\Psi_1, \ldots, \Psi_N)$ and $\Phi = \mathrm{diag}(\Phi_1, \ldots, \Phi_N)$.

5. Define $G_i := \Phi_i H_i \Psi_i$, and

$$M := \begin{bmatrix} \Psi^{-1} & 0 \\ 0 & I_d \end{bmatrix} S \begin{bmatrix} \Phi^{-1} & 0 \\ 0 & I_d \end{bmatrix}$$

Figure 2 below illustrates the interconnection whose (simplicity is now masked by the scalings) that the algorithm must attempt to certify.
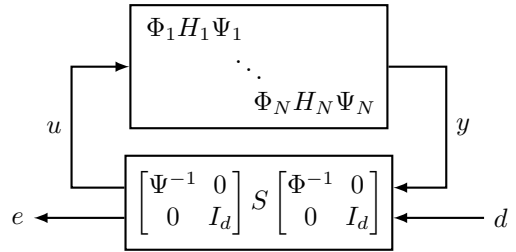

Figure 2: Scaling of the interconnected system of Figure 1 that leaves the closed-loop map unchanged.

We generated 100 random instances of the interconnected system described above, each with $N = 100$. The SOS/ADMM algorithm was used to certify the L_2-gain of the interconnected system is less than or equal to 1 for each instance. Figure 3 shows that the algorithm succeeded for all 100 tests, requiring at most 48 iterations and less than 15 iterations for 90% of the tests.
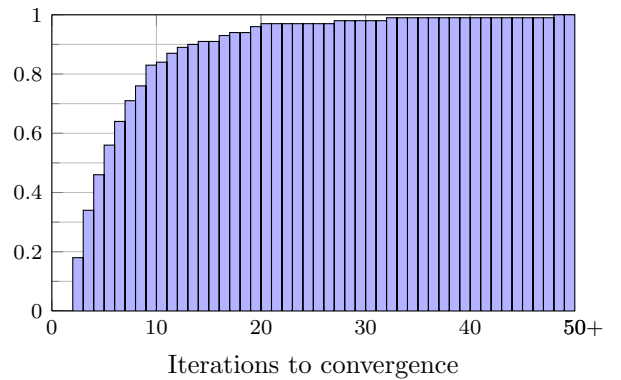

Figure 3: Cumulative plot displaying the fraction of 100 total tests that required at most a given number of iterations to certify the L_2-gain property of the interconnected system.

Vehicle platooning. In this example we analyze the L_2-gain properties of a vehicle platoon model. Each vehicle measures its distance to a subset of vehicles (indicated by dashed lines), and adjusts its throttle according

to some control law. A possible configuration is shown in Figure 4. We would like to certify that under a broad range of control law choices and measurement topologies, bounds on the L_2-gain properties can be certified. For the purpose of this example, we ignore vehicle collisions.
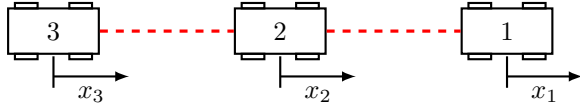

Figure 4: Vehicle platoon example. Each vehicle measures the relative distance of all vehicles connected to it by a dotted line.

Consider N vehicles in a platoon. The dynamics of the ith vehicle are described by

$$\Sigma_i : \quad \begin{aligned} \dot{v}_i &= -v_i + v_i^{\text{nom}} + u_i \\ y_i &= v_i \end{aligned} \quad i = 1, \ldots, N$$

where $v_i(t)$ is the vehicle velocity and v_i^{nom} is a nominal velocity. In the absence of a control input $u_i(t)$, each vehicle eventually tends to its nominal velocity.

Each vehicle uses the relative distance between itself and a subset of the other vehicles to control its velocity. The subsets are represented by a connected, bidirectional, acyclic graph with L links interconnecting the N vehicles. In Figure 4, the links are shown as dotted lines. Letting p_ℓ be the relative displacement between the vehicles connected by link ℓ then $\dot{p}_\ell = v_i - v_j$ where x_i is the leading node and x_j is the trailing node. We define $D \in \mathbb{R}^{N \times L}$ as

$$D_{i\ell} = \begin{cases} 1 & \text{if } i \text{ is the leading node of edge } \ell \\ -1 & \text{if } i \text{ is the trailing node of edge } \ell \\ 0 & \text{otherwise} \end{cases}$$

Thus, D maps the individual velocities of the vehicles to the relative velocity across each link. That is, $\dot{p} = D^\mathsf{T} v$.

Control strategies for velocity regulation of the platoon were presented and analyzed in [4, 5]. We will consider a general set of control strategies that encompasses those presented in [4, 5]. Specifically, we will analyze control laws of the form

$$u_i = -\sum_{\ell=1}^{L} D_{i\ell} \phi_\ell(p_\ell)$$

where $\phi_\ell : \mathbb{R} \to \mathbb{R}$ is increasing and surjective, ensuring the existence of an equilibrium point [4]. Defining $\Phi := \text{diag}(\phi_1, \ldots, \phi_L)$, we may represent the system as the block diagram in Figure 5.

The map Λ from $\dot{p}$ to $\Phi(p)$, indicated by a dashed box in Figure 5, is diagonal; each $\dot{p}_\ell$ is separately integrated and then the corresponding ϕ_ℓ is applied. Thus, we may write $\Lambda = \text{diag}(\Lambda_1, \ldots, \Lambda_L)$, where Λ_ℓ is described by

$$\Lambda_\ell : \quad \begin{aligned} \dot{p}_\ell &= \eta_\ell \\ z_\ell &= \phi_\ell(p_\ell) \end{aligned} \quad \ell = 1, \ldots, L$$
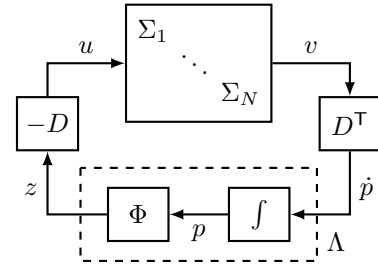

Figure 5: Block diagram of vehicle platoon dynamics.

where η_ℓ is the input and z_ℓ is the output. The interconnection structure may be transformed into the standard form of Figure 1 by diagonally concatenating $\Sigma := \text{diag}(\Sigma_1, \ldots, \Sigma_N)$ with Λ. The resulting block diagram is shown in Figure 6.
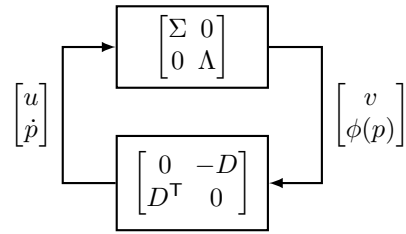

Figure 6: Transformation of Figure 5 into the standard form shown in Figure 1.

Although an equilibrium (v^*, p^*) is guaranteed to exist its location depends on the unknown functions ϕ_ℓ. Thus, we will exploit the EID properties of the subsystems to establish the desired global property without explicit knowledge of the equilibrium. For each Σ_i subsystem, we may certify EID by solving an LMI since the dynamics are linear.

For each Λ_ℓ subsystem, a SOS program to certify dissipativity would depend on the choice of the ϕ_ℓ function. However, it is not difficult to show that Λ_i is EID with respect to the following supply rate (also known as *equilibrium independent passivity*).

$$\begin{bmatrix} \eta_\ell - \eta_\ell^* \\ z_\ell - z_\ell^* \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \eta_\ell - \eta_\ell^* \\ z_\ell - z_\ell^* \end{bmatrix} \tag{19}$$

This property can be proven by using the storage function

$$V_\ell(p_\ell) = 2 \int_{p_\ell^*}^{p_\ell} [\phi_\ell(\theta) - \phi_\ell(p_\ell^*)] \, d\theta \tag{20}$$

and the property $(p_\ell - p_\ell^*)[\phi_\ell(p_\ell) - \phi_\ell(p_\ell^*)] \geq 0$ which follows because ϕ_ℓ is increasing. Therefore, instead of searching over supply rates for the Λ_ℓ subsystems in the ADMM algorithm, we fix (19) as the supply rate. Note that the ADMM algorithm does not require knowledge of the ϕ_ℓ functions or their equilibrium points; it is enough to know the supply rate (19).

For the simulation, we used $N = 20$, and each vehicle's nominal velocity was randomly chosen. A linear topology

was used as in Figure 4. That is, each vehicle measures the distance to the vehicle in front of it and the vehicle behind it. We investigated how a force disturbance applied to the trailing vehicle would affect the velocity of the lead vehicle. Specifically, we augmented the interconnection matrix M (see Figure 1) such that a disturbance d is applied to the last vehicle:

$$\dot{v}_N = -v_N + v_N^{\mathrm{nom}} + u_N + d$$

and the output e is the velocity of the first vehicle v_1. We then attempted to certify that the L_2 gain of the interconnected system from d to e is no greater than γ using a global supply rate of the form

$$\begin{bmatrix} d \\ e - e^* \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^{-2} \end{bmatrix} \begin{bmatrix} d \\ e - e^* \end{bmatrix}.$$

Using a bisection search to find the smallest certifiable γ, we found that $\gamma_{\min} \leq 0.71$. The inequality is due to the fact that our method searches over a restricted class of possible storage functions, and may therefore be conservative. In an effort to bound this conservatism, we performed an ad-hoc search over linear ϕ_ℓ functions, seeking a worst-case L_2 gain. The result was $\gamma_{\min} \geq 0.49$.

Certain provisions were made to achieve satisfactory convergence of the ADMM routine, and this highlights a potential pitfall of our method that is common to many iterative optimization algorithms. For this example, the local subproblems have the constraints $X_i^{11} \geq 0$ for $i = 1, \ldots, N$ (these are scalar variables), while the global problem has the constraint $D^{\mathsf{T}} \mathrm{diag}(X_1^{11}, \ldots, X_N^{11}) D \leq 0$. The only solution that satisfies both of these constraints is $X_i^{11} = 0$ for all i. This further implies that $X_i^{12} = X_i^{21} = 1$ for all i. Therefore, even though each of the $\mathcal{G}$ and $\mathcal{L}_i$ sets defined in (13) have a nonempty interior, their intersection does not. The result is that the ADMM algorithm oscillates between $X_i^{11} > 0$ for the local problems and $Z_i^{11} < 0$ for the global problem, leading to slow convergence and only reaching feasibility (i.e. $X_i^{11} = Z_i^{11} = 0$) in the limit. We addressed this issue by setting $X_i^{11} = 0$ and $X_i^{12} = X_i^{21} = 1$, effectively removing those variables from the optimization. The feasible region of the resulting problem has a nonempty interior, and the ADMM algorithm converged in a few iterations.

All simulations performed were implemented in MATLAB using the SOSOPT toolbox [11] to solve SOS programs and the CVX toolbox [6] to solve SDP problems.

6 Acknowledgements

References

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