Decentralized LQG Control of Systems with a Broadcast Architecture

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Several decision-makers, each with access to different subsets of the total available information

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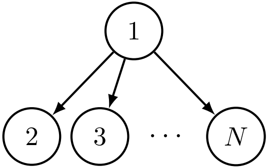
Several decision-makers, each with access to different subsets of the total available information

- Intractable in general, even in LQG setting
- Explicit solutions exist in some cases!
- Efficiently computable

Broadcast architecture (LQG)



$$\Lambda \sim \begin{bmatrix} * & 0 & \dots & 0 \\ * & * & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \dots & * \end{bmatrix}$$

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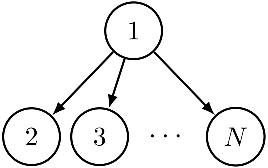

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State-space equations:

$\dot{x} = Ax + B_1 w + B_2 u$	state equation
$z = C_1 x + D_{12} u$	regulated output
$y = C_2 x + D_{21} w$	measurement

where: $A, B_i, C_i, D_{ij} \in \Lambda$ and w is white IID Gaussian noise

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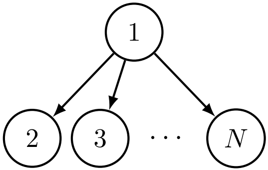

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minimize: $$\lim_{T\to\infty} \frac{1}{T} \mathbf{E} \int_0^T \left\| z(t) \right\|^2 \mathrm{d}t$$

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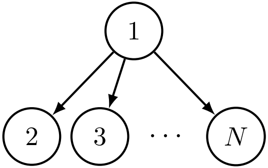

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Controller constraint:

We require that $\mathcal{K} \in \Lambda$, we assume

$$\begin{aligned} \dot{\xi} &= A_K \xi + B_K y \\ u &= C_K \xi + D_K y \end{aligned}$$

where: $A_K, B_K, C_K, D_K \in \Lambda$

Previous work

Bad news:

- Optimal controller is nonlinear even for simple LQG examples
  [Witsenhausen 1968]
- Decentralized controller synthesis is hard in general
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Good news:

- Large classes of decentralized problems can be convexified
  [Ho, Chu 1972; Qi, Salapaka, et al 2004; Rotkowitz, Lall 2006]
- In many such cases, the optimal controller is linear
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Great news:

- Explicit state-space solution for state feedback with sparsity [Swigart, Lall 2010; Shah, Parrilo 2010]
- General output feedback solved for 2-player $(N = 2)$ case [Lessard, Lall 2011]

Outline

- classical centralized case $(N = 1)$
- solution for the broadcast case (any N)
- outline of method
- numerical example

Centralized case

$$
\begin{aligned}
\dot{x} &= Ax + B_1 w + B_2 u \\
z &= C_1 x + D_{12} u \\
y &= C_2 x + D_{21} w
\end{aligned}
$$

no structural constraint on $\mathcal{K}$

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Optimal Controller

$$\dot{\xi} = A\xi + B_2 u - L(y - C_2\xi)$$
$$u = K\xi$$

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- Kalman filter: ξ is the steady-state estimate of x given y.
- LQR static controller: $u = K\xi$

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- K and L come from solving standard AREs

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Example (3 nodes)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & 0 & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} B_{11} & 0 & 0 \\ B_{21} & B_{22} & 0 \\ B_{31} & 0 & B_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} C_{11} & 0 & 0 \\ C_{21} & C_{22} & 0 \\ C_{31} & 0 & C_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

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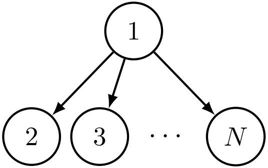
- u_1 measures y_1
- u_2 measures y_1 and y_2
- u_3 measures y_1 and y_3

Explicit solution



Optimal controller (root node)

$$\begin{aligned}\dot{\zeta} &= A\zeta + B\hat{u} - \hat{L}(y_1 - C_{11}\zeta_1)\\ \hat{u} &= K\zeta\\ u_1 &= \hat{u}_1\end{aligned}$$

Optimal controller (leaf nodes)

$$\dot{\xi}^k = \begin{bmatrix} A_{11} & 0 \\ A_{k1} & A_{kk} \end{bmatrix} \xi^k + \begin{bmatrix} B_{11} & 0 \\ B_{k1} & B_{kk} \end{bmatrix} \begin{bmatrix} u_1 \\ u_k \end{bmatrix} - L^{1k}\left(\begin{bmatrix} y_1 \\ y_k \end{bmatrix} - \begin{bmatrix} C_{11} & 0 \\ C_{k1} & C_{kk} \end{bmatrix} \xi^k \right)$$

$$u_k = \hat{u}_k + \hat{K}_k\left(\xi^k - \begin{bmatrix} \zeta_1 \\ \zeta_k \end{bmatrix}\right)$$

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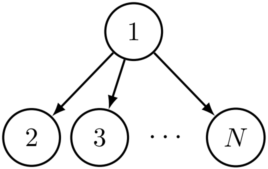

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- K and L^{1k} come from solving standard AREs

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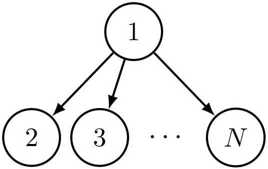

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- K and L^{1k} come from solving standard AREs
- $\hat{L}$ and $\hat{K}_k$ can be computed efficiently

Solution structure

$$\begin{aligned} \dot{\zeta} &= A\zeta + B\hat{u} - \hat{L}\big(y_1 - C_{11}\zeta_1\big) \\ \hat{u} &= K\zeta \end{aligned}$$

(root node)

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(for $k = 2, \dots, N$)

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Structural result

- ζ and ξ^k are optimal steady-state estimators

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- ζ and ξ^k are optimal steady-state estimators
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- ξ^k estimates $\begin{bmatrix} x_1 \\ x_k \end{bmatrix}$ given $\begin{bmatrix} y_1 \\ y_k \end{bmatrix}$ for $k = 2, \dots, N$

Outline of solution (stable case)

1. Write in input-output form

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$u = \mathcal{K} y$$

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2. Convert to a model-matching problem

$$\underset{\substack{\mathcal{K} \text{ stabilizing} \\ \mathcal{K} \in \Lambda}}{\text{minimize}} \left\| \mathcal{P}_{11} + \mathcal{P}_{12} \mathcal{K} (I - \mathcal{P}_{22} \mathcal{K})^{-1} \mathcal{P}_{21} \right\|$$

becomes: $\underset{\substack{\mathcal{Q} \text{ stable} \\ \mathcal{Q} \in \Lambda}}{\text{minimize}} \left\| \mathcal{T}_1 + \mathcal{T}_2 \mathcal{Q} \mathcal{T}_3 \right\|$

Outline of solution (stable case)

3. Person-by-person optimality

$$\underset{Q_{ij}\ \text{stable}}{\text{minimize}} \left\| \mathcal{T}_1 + \mathcal{T}_2 \begin{bmatrix} \mathcal{Q}_{11} & 0 & \dots & 0 \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Q}_{N1} & 0 & \dots & \mathcal{Q}_{NN} \end{bmatrix} \mathcal{T}_3 \right\|$$

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5. Transform back $\mathcal{Q} \to \mathcal{K}$.

Numerical example

Randomly generated structured plant, $N = 3$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \left[\begin{array}{cc:c:cc} -3.7 & -4.6 & 0 & 0 & 0 \\ 4.6 & -3.7 & 0 & 0 & 0 \\ \hdashline 0.10 & 0.085 & -2.5 & 0 & 0 \\ \hdashline 0.86 & 0.13 & 0 & -0.40 & -0.078 \\ 0.32 & 0.78 & 0 & -0.078 & -1.1 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \left[\begin{array}{c:c:cc} 0.60 & 0 & 0 & 0 \\ 0.95 & 0 & 0 & 0 \\ \hdashline 0.063 & 0.89 & 0 & 0 \\ \hdashline 0.016 & 0 & 0.21 & 0.48 \\ 0.80 & 0 & 0.67 & 0.64 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \left[\begin{array}{cc:c:cc} 0.049 & 0.17 & 0 & 0 & 0 \\ 0.026 & 0.61 & 0 & 0 & 0 \\ \hdashline 0.86 & 0.93 & 0.16 & 0 & 0 \\ \hdashline 0.21 & 0.72 & 0 & 0.21 & 0.75 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$W = \left[\begin{array}{cc:c:cc} 2.3 & 1.4 & 1.8 & 1.7 & 1.9 \\ 1.4 & 1.3 & 1.3 & 1.7 & 1.5 \\ \hdashline 1.8 & 1.3 & 3.7 & 1.4 & 1.7 \\ \hdashline 1.7 & 1.7 & 1.4 & 3.7 & 2.5 \\ 1.9 & 1.5 & 1.7 & 2.5 & 3.0 \end{array}\right] \quad V = \left[\begin{array}{cc:c:c} 1.8 & 1.1 & 1.1 & 0.59 \\ 1.1 & 1.5 & 1.3 & 0.84 \\ \hdashline 1.1 & 1.3 & 2.9 & 0.86 \\ \hdashline 0.59 & 0.84 & 0.86 & 2.7 \end{array}\right] \quad U = \left[\begin{array}{cc:c:c} 1.0 & 1.4 & 1.4 & 1.4 \\ 1.2 & 1.0 & 0.83 & 0.89 \\ \hdashline 1.4 & 1.7 & 2.2 & 0.94 \\ \hdashline 1.4 & 1.0 & 0.77 & 2.1 \\ 1.3 & 1.4 & 1.2 & 2.3 \end{array}\right]$$

$$Q = \left[\begin{array}{cc:c:cc} 6.4 & 4.8 & 2.2 & 2.2 & 1.5 \\ 4.8 & 7.0 & 1.3 & 2.5 & 1.8 \\ \hdashline 2.2 & 1.3 & 2.7 & 0 & 0 \\ \hdashline 2.2 & 2.5 & 0 & 2.6 & 1.9 \\ 1.5 & 1.8 & 0 & 1.9 & 1.6 \end{array}\right] \quad R = \left[\begin{array}{c:c:cc} 8.2 & 2.0 & 2.1 & 1.8 \\ \hdashline 2.0 & 2.7 & 0 & 0 \\ \hdashline 2.1 & 0 & 2.0 & 1.8 \\ 1.8 & 0 & 1.8 & 2.2 \end{array}\right] \quad S = \left[\begin{array}{c:c:cc} 6.1 & 2.3 & 2.0 & 1.8 \\ 5.9 & 1.5 & 2.1 & 2.5 \\ \hdashline 1.7 & 2.5 & 0 & 0 \\ \hdashline 2.5 & 0 & 1.8 & 2.1 \\ 1.6 & 0 & 1.2 & 1.5 \end{array}\right]$$

Numerical example

$$A_K = \left[\begin{array}{ccccc|ccc|cccc}
-4.0 & -5.6 & 0.012 & -0.083 & -0.056 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4.0 & -4.6 & 0.019 & -0.13 & -0.088 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.36 & -0.66 & -3.3 & 0.078 & 0.053 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.68 & -0.6 & -0.0028 & -0.87 & -0.48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.44 & -0.54 & 0.0013 & -0.75 & -1.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-0.33 & -0.4 & 0.012 & -0.083 & -0.056 & -3.8 & -5.3 & -0.026 & 0 & 0 & 0 & 0 \\
-0.52 & -0.63 & 0.019 & -0.13 & -0.088 & 4.6 & -4.0 & 0.0038 & 0 & 0 & 0 & 0 \\
0.31 & 0.38 & -0.011 & 0.078 & 0.053 & -1.0 & -1.2 & -3.4 & 0 & 0 & 0 & 0 \\
\hline
-0.33 & -0.4 & 0.012 & -0.083 & -0.056 & 0 & 0 & 0 & -3.8 & -5.3 & -0.058 & -0.21 \\
-0.52 & -0.63 & 0.019 & -0.13 & -0.088 & 0 & 0 & 0 & 4.5 & -4.1 & -0.039 & -0.14 \\
0.077 & 0.073 & -0.0028 & 0.023 & 0.013 & 0 & 0 & 0 & 0.44 & -0.95 & -1.1 & -1.1 \\
-0.034 & -0.059 & 0.0013 & -0.0065 & -0.0055 & 0 & 0 & 0 & -0.56 & -0.73 & -0.89 & -2.1
\end{array}\right]$$

$$B_K = \left[\begin{array}{cc|c|c}
-0.026 & 0.97 & 0 & 0 \\
0.46 & 0.37 & 0 & 0 \\
0.16 & 1.0 & 0 & 0 \\
0.62 & 0.28 & 0 & 0 \\
0.25 & 0.8 & 0 & 0 \\
\hline
-0.05 & 0.85 & 0.16 & 0 \\
0.47 & 0.38 & -0.024 & 0 \\
0.094 & 0.69 & 0.42 & 0 \\
\hline
-0.015 & 0.81 & 0 & 0.28 \\
0.47 & 0.25 & 0 & 0.18 \\
0.67 & -0.26 & 0 & 0.83 \\
0.29 & 0.33 & 0 & 0.74
\end{array}\right]
\qquad
\mathcal{K}(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & 0 \end{array}\right]$$

$$C_K = \left[\begin{array}{ccccc|ccc|cccc}
-0.55 & -0.66 & 0.02 & -0.14 & -0.092 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0.39 & 0.48 & -0.014 & 0.098 & 0.066 & -0.83 & -0.54 & -0.94 & 0 & 0 & 0 & 0 \\
\hline
0.73 & 0.91 & -0.026 & 0.18 & 0.12 & 0 & 0 & 0 & -1.0 & -0.11 & -0.0084 & 0.26 \\
-0.14 & -0.23 & 0.0051 & -0.025 & -0.023 & 0 & 0 & 0 & 0.02 & -1.0 & -1.0 & -0.98
\end{array}\right]$$

Numerical example

control structure	state dimension	optimal cost
open-loop	0	5.5928
$\begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & 0 & * \end{bmatrix}$	$3n_1 + 2n_2 + 2n_3$	2.8240
$\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$	$2n_1 + 2n_2 + 2n_3$	2.8111
$\begin{bmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$	$2n_1 + 2n_2 + 2n_3$	2.8044
$\begin{bmatrix} * & 0 & * \\ * & * & 0 \\ * & * & * \end{bmatrix}$	$2n_1 + 2n_2 + 2n_3$	2.7572
$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$n_1 + n_2 + n_3$	2.7503

Summary

For broadcast output feedback:

- We know the optimal state dimension
- Optimal controller can be computed *efficiently*
  - Some gains are decoupled, found by solving separate AREs
  - Others are coupled, can be computed efficiently
- Structure: each player estimates part of the global state

Thank you!

Papers and additional information available at:
www.laurentlessard.com