A State-Space Solution to the Two-Player Decentralized Optimal Control Problem

Laurent Lessard[1,3] Sanjay Lall[2,3]

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Abstract

In this paper, we present an explicit state-space solution to the two-player decentralized optimal control problem. In this problem, there are two interconnected linear systems that seek to optimize a global quadratic cost. Both controllers perform output feedback, but they have access to different subsets of the available measurements. The optimal controller, which was not previously known, has a state dimension equal to twice the state dimension of the original system.

1 Introduction

In this paper, we address optimal controller synthesis for the decentralized two-player problem. The feature that makes this problem difficult is the structural constraint imposed on the controller. Such constraints appear frequently in practice. For example, many modular systems such as power grids, or teams of vehicles flying in formation, can be viewed as a network of interconnected subsystems. A common feature of these applications is that subsystems must make control decisions with limited information. The goal is to optimize global performance measures despite the decentralized nature of the system.

In this paper, we consider a specific information structure in which there are two linear subsystems and the state-space matrices are block-triangular:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A_{11} & 0\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}B_{11} & 0\\ B_{21} & B_{22}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} + w$$
$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}C_{11} & 0\\ C_{21} & C_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + v$$

In other words, Player 1's measurements and dynamics only depend on Player 1's inputs, but Player 2's system is fully coupled. Our aim is to find an output-feedback law with this same structure; u_1 must depend only on y_1, but u_2 is allowed to depend on both y_1 and y_2.

[1]L. Lessard is with the Department of Aeronautics and Astronautics at Stanford University, Stanford, CA 94305, USA. `lessard@stanford.edu`

[2]S. Lall is with the Department of Electrical Engineering, and the Department of Aeronautics and Astronautics at Stanford University, Stanford, CA 94305, USA. `lall@stanford.edu`

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The controller must be stabilizing, and must also minimize the infinite-horizon quadratic cost

$$\lim_{T\to\infty}\frac{1}{T}\,\mathbb{E}\int_0^T\Big(x(t)^\mathsf{T}Qx(t) + u(t)^\mathsf{T}Ru(t)\Big)\,\mathrm{d}t$$

The disturbance w and noise v are assumed to be stationary zero-mean Gaussian processes, and are characterized by the covariace matrices

$$\operatorname{cov} w = W \quad\text{and}\quad \operatorname{cov} v = V$$

The structural constraint that u_1 and u_2 depend on different sets of measurements greatly complicates the problem. It was shown by Witsenhausen [12] that such structural constraints can lead to situations in which a nonlinear control policy strictly outperforms any linear policy. However, this is not always the case. For many decentralized problems, there exists a linear optimal policy, and it can be found by solving a convex optimization problem [2, 3, 4, 11]. The two-player problem considered here falls in this category, so we may restrict our search to linear controllers.

In [5], it was shown that certain decentralized problems can be converted to equivalent centralized problems via vectorization. Consequently, the optimal controller for the two-player problem must be rational and we may further restrict our search to controllers representable in state-space form. Unfortunately, vectorization causes a dramatic increase in the state dimension of the system, making it a feasible method only for small problems.

Explicit solutions have been found, but only for some special cases of the two-player problem. Most notably, the state-feedback case admits a nice state-space solution [8, 9]. More recently, the partial output feedback case was also solved, in which the second player performs output feedback but the first player performs state-feedback and provides his full state to the second player [10].

In this paper, we provide explicit state-space formulae for an optimal controller. These formulae provide tight upper bounds on the minimal state dimension for an optimal controller, which were previously not known. The paper is organized as follows. In Section 2, we review some required background mathematics and notation. In Section 3, we formulate the problem as an $\mathcal{H}_2$ optimization. In Section 4, we present our main results, and in subsequent sections, we discuss the result and give a abridged proof.

2 Preliminaries

State-space. In this paper, all systems are linear and time-invariant (LTI), rational, and continuous-time. Given a state-space representation (A, B, C, D) for such a system, we can describe the input-output map as a matrix of proper rational functions

$$\mathcal{F} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \triangleq D + C(sI - A)^{-1}B$$

If the realization is minimal, $\mathcal{F}$ having stable poles is equivalent to A being Hurwitz, and $\mathcal{F}$ being strictly proper is equivalent to $D = 0$.

Sylvester Equations. A Sylvester equation is a matrix equation of the form

$$AX + XB + C = 0$$

where A and B are square matrices, possibly of different sizes. Here, we must solve for X and all other parameters are known. We write $X = \mathrm{LYAP}(A, B, C)$ to denote a solution when it exists.

Riccati Equations. A continuous-time algebraic Riccati equation (CARE) is a matrix equation of the form

$$A^\mathsf{T} X + XA - XBR^{-1}B^\mathsf{T} X + Q = 0$$

Again, we must solve for X and all other parameters are known. We say X is a *stabilizing* solution if $(A + BK)$ is stable, where $K = -R^{-1}B^\mathsf{T} X$ is the associated gain matrix. We write $X = \mathrm{CARE}(A, B, Q, R)$ to denote a stabilizing solution when it exists.

Stabilization. For simplicity, we assume throughout this paper that the plant dynamics are stable. No generality is lost in this assumption because a parameterization of all stabilizing controllers was found for many decentralized problems including the two-player problem [6, 7]. This leads to a coprime factorization that preserves the triangular structure of problem and is similar to the classical treatment for centralized problems [13].

3 Problem Formulation

In this section, we will formulate the problem of Section 1 as an $\mathcal{H}_2$ optimal control problem using the language of transfer functions. The state-space equations are

$$\dot{x} = Ax + Bu + Mw, \tag{1}$$

$$z = Fx + Hu \tag{2}$$

$$y = Cx + Nw \tag{3}$$

Our goal is to find a LTI controller $\mathcal{K}$ that maps y to u, and minimizes the average infinite-horizon cost $\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \|z(t)\|^2\,\mathrm{d}t$.

We have some additional structure:

$$A \triangleq \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad B \triangleq \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \quad C \triangleq \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix}$$

and we impose a similar structure on our controller $\mathcal{K}$. We denote the set of block lower-triangular operators as S, and omit the specific class of operators from this notation for convenience. We therefore write the constraint as $\mathcal{K} \in S$. To ease notation, define

$$E_1 \triangleq \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \text{and} \quad E_2 \triangleq \begin{bmatrix} 0 \\ I \end{bmatrix}$$

where sizes of the identity matrices involved are determined by context. We also partition B by its block-columns and C by its block-rows. Thus, $B_1 \triangleq BE_1$, $B_2 \triangleq BE_2$, $C_1 \triangleq E_1^\mathsf{T} C$, and $C_2 \triangleq E_2^\mathsf{T} C$. For consistency with Section 1, suppose $F^\mathsf{T} H = 0$ and $MN^\mathsf{T} = 0$, and

$$Q \triangleq F^\mathsf{T} F \quad R \triangleq H^\mathsf{T} H \quad W \triangleq MM^\mathsf{T} \quad V \triangleq NN^\mathsf{T}$$

As is standard, we assume $R > 0$ and $V > 0$ so that the problem is nonsingular. By taking Laplace transforms of (1)–(3), and eliminating x,

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \tag{4}$$

where the $\mathcal{P}_{ij}$ are transfer functions given by

$$\begin{aligned} \mathcal{P}_{11} &= \left[\begin{array}{c|c} A & M \\ \hline F & 0 \end{array}\right] \quad & \mathcal{P}_{12} &= \left[\begin{array}{c|c} A & B \\ \hline F & H \end{array}\right] \\ \mathcal{P}_{21} &= \left[\begin{array}{c|c} A & M \\ \hline C & N \end{array}\right] \quad & \mathcal{P}_{22} &= \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \end{aligned} \tag{5}$$

As mentioned previously, we assume the $\mathcal{P}_{ij}$ are stable, so A is Hurwitz. Substituting $u = \mathcal{K}y$ and eliminating y and u from (4), we obtain the closed-loop map

$$z = \left(\mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21}\right) w \tag{6}$$

Since minimizing the average infinite-horizon cost is equivalent to minimizing the $\mathcal{H}_2$-norm of the closed-loop map, we seek to

$$\begin{aligned} \text{minimize} \quad & \left\|\mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21}\right\| \\ \text{subject to} \quad & \mathcal{K} \text{ is proper and rational} \\ & \mathcal{K} \text{ is stabilizing} \\ & \mathcal{K} \in S \end{aligned} \tag{7}$$

For more details on the $\mathcal{H}_2$ norm and related concepts see [1, 14]. Now let $\mathcal{Q} = \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}$, which corresponds to the well-known Youla parameterization [13] in the centralized case. Since $\mathcal{P}_{22} \in S$ and $\mathcal{K} \in S$, we have $\mathcal{Q} \in S$. Furthermore, $\mathcal{P}_{22}$ is stable and strictly proper, and $\mathcal{K}$ is stabilizing and proper. Thus, $\mathcal{Q}$ is stable and proper. The assumptions that $R > 0$ and $V > 0$ further imply that $\mathcal{Q}$ must be strictly proper to ensure finiteness of the norm. We would therefore like to solve

$$\begin{aligned} &\text{minimize} && \left\|\mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21}\right\| \\ &\text{subject to} && \mathcal{Q} \in \mathcal{RH}_2 \cap S \end{aligned} \tag{8}$$

where $\mathcal{RH}_2$ is the set of strictly proper rational functions with stable poles. Note that the $\mathcal{Q}$-substitution is invertible, and its inverse is $\mathcal{K} = \mathcal{Q}(I + \mathcal{P}_{22}\mathcal{Q})^{-1}$. So solving (8) will give us a solution to the original problem (7).

4 Main Results

In this section, we present our main result: explicit solutions to (7) and (8). We begin with some assumptions.

A1. $R > 0$ and $V > 0$

A2. (A, B_2) and (A, W) are controllable

A3. (C_1, A) and (Q, A) are observable

Next, we present the equations we will need to solve in order to construct the optimal controller. First, we have two control CAREs and their associated gains

$$\begin{aligned} X &= \text{CARE}(A, B, Q, R) \\ K &= -R^{-1}B^{\mathsf{T}}X \end{aligned} \tag{9}$$

$$\begin{aligned} \tilde{X} &= \text{CARE}(A, B_2, Q, R_{22}) \\ \tilde{K} &= -R_{22}^{-1}B_2^{\mathsf{T}}\tilde{X} = \begin{bmatrix} \tilde{K}_1 & \tilde{K}_2 \end{bmatrix} \end{aligned} \tag{10}$$

Next, we have the analogous set of estimation equations.

$$\begin{aligned} Y &= \text{CARE}(A^{\mathsf{T}}, C^{\mathsf{T}}, W, V) \\ L &= -YC^{\mathsf{T}}V^{-1} \end{aligned} \tag{11}$$

$$\begin{aligned} \tilde{Y} &= \text{CARE}(A^{\mathsf{T}}, C_1^{\mathsf{T}}, W, V_{11}) \\ \tilde{L} &= -\tilde{Y}C_1^{\mathsf{T}}V_{11}^{-1} = \begin{bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{bmatrix} \end{aligned} \tag{12}$$

Finally, we define a pair of coupled linear equations that must also be solved for Φ and Ψ.

$$\begin{aligned} &(A_{22} + B_{22}\tilde{K}_2)^{\mathsf{T}}\Phi + \Phi(A_{11} + \tilde{L}_1C_{11}) \\ &\qquad + E_2^{\mathsf{T}}(\tilde{X} - X)(\tilde{L} - E_2\Psi C_{11}^{\mathsf{T}}V_{11}^{-1})C_{11} = 0 \\ &(A_{22} + B_{22}\tilde{K}_2)\Psi + \Psi(A_{11} + \tilde{L}_1C_{11})^{\mathsf{T}} \\ &\qquad + B_{22}(\tilde{K} - R_{22}^{-1}B_{22}^{\mathsf{T}}\Phi E_1^{\mathsf{T}})(\tilde{Y} - Y)E_1 = 0 \end{aligned} \tag{13}$$

Note that these equations are linear in Φ and Ψ and can be solved easily; for example, they may be written in standard $Ax = b$ form using the Kronecker product. Finally, we define two new gains:

$$\begin{aligned} \hat{K} &= \tilde{K} - R_{22}^{-1}B_{22}^{\mathsf{T}}\Phi E_1^{\mathsf{T}} \\ \hat{L} &= \tilde{L} - E_2\Psi C_{11}^{\mathsf{T}}V_{11}^{-1} \end{aligned} \tag{14}$$

What follows are the main results of the paper.

Theorem 1. *Suppose assumptions A1–A3 hold. Then* (9)–(12) *have stabilizing solutions,* (13) *has a unique solution, and an optimal solution to* (8) *is given by*

$$\mathcal{Q}_{opt} = \left[\begin{array}{ccc|c} A + BK & \hat{L}C_1 & 0 & -\hat{L}E_1^{\mathsf{T}} \\ 0 & A + B_2\hat{K} + \hat{L}C_1 & -B_2\hat{K} & -\hat{L}E_1^{\mathsf{T}} \\ 0 & 0 & A + LC & -L \\ \hline K & -E_2\hat{K} & E_2\hat{K} & 0 \end{array}\right] \tag{15}$$

Theorem 2. *Suppose assumptions A1–A3 hold. An optimal solution to* (7) *is given by*

$$\mathcal{K}_{opt} = \left[\begin{array}{cc|c} A + BK + \hat{L}C_1 & 0 & -\hat{L}E_1^{\mathsf{T}} \\ BK - B_2\hat{K} & A + LC + B_2\hat{K} & -L \\ \hline K - E_2\hat{K} & E_2\hat{K} & 0 \end{array}\right] \tag{16}$$

5 State Dimension

First, note that $\mathcal{Q}_{\text{opt}}$ and $\mathcal{K}_{\text{opt}}$ have the correct block-triangular structure. We can also verify that $\mathcal{Q}_{\text{opt}}$ is stable; the eigenvalues of its A-matrix are the eigenvalues of $A + BK$, $A_{11} + \tilde{L}_1C_{11}$, and $A_{22} + B_{22}\tilde{K}_2$, and $A + LC$ which are stable by construction.

The formula (16) gives an upper bound on the state dimension of the optimal controller. If $A_{11} \in \mathbb{R}^{n_1 \times n_1}$ and $A_{22} \in \mathbb{R}^{n_2 \times n_2}$, then $\mathcal{K}_{\text{opt}}$ has at most $2n_1 + 2n_2$ states. However, notice that this number may be different in a decentralized implementation. In particular, if the two controllers do not communicate with one another, the first controller needs a realization of $\mathcal{K}_{11}$ while the second controller needs a realization of $\begin{bmatrix} \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix}$. In this case, the first controller will have $n_1 + n_2$ states, and the second controller will have $2n_1 + 2n_2$ states. Note as well that if we make the problem centralized by removing the structural constraint on the controller, the optimal controller requires $n_1 + n_2$ states.

6 Proof of Main Results

Our general technique is to separate the decentralized problem (8) into two coupled centralized problems. Through further manipulations, we solve the resulting set of coupled optimality conditions. To this end, we will need the solution of a general type of $\mathcal{H}_2$ optimization problem

$$\begin{aligned} &\text{minimize} && \|\mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21}\| \\ &\text{subject to} && \mathcal{Q} \in \mathcal{RH}_2 \end{aligned} \tag{17}$$

but unlike (8), the $\mathcal{P}_{ij}$ have different A matrices in their state-space realizations.

Lemma 3. *Suppose $\mathcal{P}_{11}$, $\mathcal{P}_{12}$, and $\mathcal{P}_{21}$ are matrices of stable transfer functions with state-space realizations*

$$\mathcal{P}_{11} = \left[\begin{array}{c|c} A & J \\ \hline G & 0 \end{array}\right] \quad \mathcal{P}_{12} = \left[\begin{array}{c|c} \tilde{A} & B \\ \hline F & H \end{array}\right] \quad \mathcal{P}_{21} = \left[\begin{array}{c|c} \hat{A} & M \\ \hline C & N \end{array}\right]$$

Note that A, $\tilde{A}$, and $\hat{A}$ may be different matrices. Suppose there exists stabilizing solutions to the CAREs

$$\begin{aligned} X &= \text{CARE}(\tilde{A}, B, Q, R), & K &= -R^{-1}B^{\mathsf{T}}X \\ Y &= \text{CARE}(\hat{A}^{\mathsf{T}}, C^{\mathsf{T}}, W, V), & L &= -YC^{\mathsf{T}}V^{-1} \end{aligned}$$

Then, there exists unique solutions to the equations

$$\begin{aligned} \tilde{Z} &= \text{LYAP}\left((\tilde{A}+BK)^{\mathsf{T}}, A, (F+HK)^{\mathsf{T}}G\right) \\ \hat{Z} &= \text{LYAP}\left(A, (\hat{A}+LC)^{\mathsf{T}}, J(M+LN)^{\mathsf{T}}\right) \end{aligned} \tag{18}$$

Furthermore, a solution to (17) is given by

$$\mathcal{Q}_{opt} = -\mathcal{W}_L^{-1}\left[\begin{array}{c|c} A & JN^{\mathsf{T}} + \hat{Z}C^{\mathsf{T}} \\ \hline B^{\mathsf{T}}\tilde{Z} + H^{\mathsf{T}}G & 0 \end{array}\right]\mathcal{W}_R^{-1} \tag{19}$$

where $\mathcal{W}_L$ and $\mathcal{W}_R$ are defined by

$$\mathcal{W}_L = \left[\begin{array}{c|c} \tilde{A} & B \\ \hline -R^{1/2}K & R^{1/2} \end{array}\right] \quad \mathcal{W}_R = \left[\begin{array}{c|c} \hat{A} & -LV^{1/2} \\ \hline C & V^{1/2} \end{array}\right]$$

Proof. This problem is called the $\mathcal{H}_2$ model-matching problem, and can be solved by writing out the optimality condition and using spectral factorization techniques. See for example [14, §13]. ∎

The observation that allows us to separate the decentralized problem into two centralized problems is as follows: if we assume $\mathcal{Q}_{11}$ is known, and possibly suboptimal, then the problem of finding the optimal $\begin{bmatrix}\mathcal{Q}_{21} & \mathcal{Q}_{22}\end{bmatrix}$ is centralized:

$$\begin{aligned} \min \quad & \left\|(\mathcal{P}_{11}+\mathcal{P}_{12}E_1\mathcal{Q}_{11}E_1^{\mathsf{T}}\mathcal{P}_{21})+\mathcal{P}_{12}E_2\begin{bmatrix}\mathcal{Q}_{21} & \mathcal{Q}_{22}\end{bmatrix}\mathcal{P}_{21}\right\| \\ \text{s.t.} \quad & \begin{bmatrix}\mathcal{Q}_{21} & \mathcal{Q}_{22}\end{bmatrix} \in \mathcal{RH}_2 \end{aligned} \tag{20}$$

Similarly, we may fix $\mathcal{Q}_{22}$. Our centralized optimization problem is then:

$$\begin{aligned} \min \quad & \left\|\left(\mathcal{P}_{11}+\mathcal{P}_{12}E_2\mathcal{Q}_{22}E_2^{\mathsf{T}}\mathcal{P}_{21}\right)+\mathcal{P}_{12}\begin{bmatrix}\mathcal{Q}_{11} \\ \mathcal{Q}_{21}\end{bmatrix}E_1^{\mathsf{T}}\mathcal{P}_{21}\right\| \\ \text{s.t.} \quad & \begin{bmatrix}\mathcal{Q}_{11} \\ \mathcal{Q}_{21}\end{bmatrix} \in \mathcal{RH}_2 \end{aligned} \tag{21}$$

Lemmas 4 and 5 give the solutions to (20) and (21), respectively.

Lemma 4. *Suppose $\mathcal{Q}_{11} \in \mathcal{RH}_2$ and has a realization*

$$\mathcal{Q}_{11} = \left[\begin{array}{c|c} A_P & B_P \\ \hline C_P & 0 \end{array}\right]$$

Suppose that stabilizing solutions exist to the CAREs

$$\begin{aligned} Y &= \text{CARE}(A^{\mathsf{T}}, C^{\mathsf{T}}, W, V), \quad L = -YC^{\mathsf{T}}V^{-1} \\ \tilde{X} &= \text{CARE}(A, B_2, Q, R_{22}), \\ & \qquad \tilde{K} = -R_{22}^{-1}B_2^{\mathsf{T}}\tilde{X} = \begin{bmatrix}\tilde{K}_1 & \tilde{K}_2\end{bmatrix} \end{aligned}$$

Then there exists a unique solution to the equation

$$\begin{bmatrix}\Phi & \tilde{Z}_3\end{bmatrix} = \text{LYAP}\left((A_{22}+B_{22}\tilde{K}_2)^{\mathsf{T}}, \begin{bmatrix}A_{11} & 0 \\ B_PC_{11} & A_P\end{bmatrix}, \begin{bmatrix}0 & (E_2^{\mathsf{T}}\tilde{X}B_1+\tilde{K}_2^{\mathsf{T}}R_{21})C_p\end{bmatrix}\right) \tag{22}$$

Furthermore, a solution to (20) is given by

$$\begin{bmatrix}\mathcal{Q}_{21} & \mathcal{Q}_{22}\end{bmatrix}_{opt} = \left[\begin{array}{c|c} A_{22}+B_{22}\tilde{K}_2 & B_{22} \\ \hline \tilde{K}_2 & I \end{array}\right] \times \left[\begin{array}{ccc|c} A & 0 & B_1C_P & 0 \\ 0 & A+LC & 0 & -L \\ 0 & 0 & A_P & B_PE_1^{\mathsf{T}} \\ \hline \tilde{K} & \hat{K} & -R_{22}^{-1}(B_{22}^{\mathsf{T}}\tilde{Z}_3+R_{21}C_P) & 0 \end{array}\right] \tag{23}$$

where we have defined $\hat{K} = \tilde{K} - R_{22}^{-1}B_{22}^{\mathsf{T}}\Phi E_1^{\mathsf{T}}$.

Proof. The components of (20) may be simplified. Routine algebraic manipulations yield

$$\mathcal{P}_{11}+\mathcal{P}_{12}E_1\mathcal{Q}_{11}E_1^{\mathsf{T}}\mathcal{P}_{21} = \left[\begin{array}{ccc|c} A & 0 & B_1C_P & 0 \\ 0 & A & 0 & M \\ 0 & B_pC_1 & A_P & B_PE_1^{\mathsf{T}}N \\ \hline F & F & HE_1C_P & 0 \end{array}\right]$$

and

$$\mathcal{P}_{12}E_2 = \left[\begin{array}{c|c} A_{22} & B_{22} \\ \hline FE_2 & HE_2 \end{array}\right]$$

Since (20) is centralized, we may apply Lemma 3, and the optimal $\begin{bmatrix}\mathcal{Q}_{21} & \mathcal{Q}_{22}\end{bmatrix}$ is given by (19). This formula can be simplified considerably if we take a closer look at the Sylvester equations (18). The estimation equation,

$$\hat{Z} = \text{LYAP}\left(\begin{bmatrix}A & 0 & B_1C_P \\ 0 & A & 0 \\ 0 & B_pC_1 & A_P\end{bmatrix}, (A+LC)^{\mathsf{T}}, \begin{bmatrix}0 \\ W \\ B_PE_1^{\mathsf{T}}VL^{\mathsf{T}}\end{bmatrix}\right)$$

is satisfied by $\hat{Z} = \begin{bmatrix}0 & Y & 0\end{bmatrix}^{\mathsf{T}}$, which does not depend on A_P, B_P, or C_P. The control equation

$$\tilde{Z} = \text{LYAP}\left((A_{22}+B_{22}\tilde{K}_2)^{\mathsf{T}}, \begin{bmatrix}A & 0 & B_1C_P \\ 0 & A & 0 \\ 0 & B_PC_1 & A_P\end{bmatrix}, \begin{bmatrix}E_2^{\mathsf{T}}Q & E_2^{\mathsf{T}}Q & \tilde{K}_2^{\mathsf{T}}R_{21}C_P\end{bmatrix}\right)$$

has a first subequation that decouples from the rest, and whose solution is $E_2^\mathsf{T}\tilde{X}$. Indeed, $\tilde{Z}$ must be of the form:

$$\tilde{Z} = \begin{bmatrix} E_2^\mathsf{T}\tilde{X} & E_2^\mathsf{T}\tilde{X} + \Phi E_1^\mathsf{T} & \tilde{Z}_3 \end{bmatrix}$$

where Φ and $\tilde{Z}_3$ satisfy (22). Substituting into (19) and simplifying, we obtain (23). ∎

Lemma 5. *Suppose $\mathcal{Q}_{22} \in \mathcal{RH}_2$ and has a realization*

$$\mathcal{Q}_{22} = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & 0 \end{array}\right]$$

Suppose that stabilizing solutions exist to the CAREs

$$X = \mathrm{CARE}(A, B, Q, R), \quad K = -R^{-1}B^\mathsf{T}X$$
$$\tilde{Y} = \mathrm{CARE}(A^\mathsf{T}, C_1^\mathsf{T}, W, V_{11}),$$
$$\tilde{L} = -\tilde{Y}C_1^\mathsf{T}V_{11}^{-1} = \begin{bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{bmatrix}$$

Then there exists a unique solution to the equation

$$\begin{bmatrix} \Psi \\ \hat{Z}_3 \end{bmatrix} = \mathrm{LYAP}\left(\begin{bmatrix} A_{22} & B_{22}C_Q \\ 0 & A_Q \end{bmatrix}, (A_{11} + \tilde{L}_1 C_{11})^\mathsf{T}, \begin{bmatrix} 0 \\ B_Q(C_2\tilde{Y}E_1 + V_{21}\tilde{L}_1^\mathsf{T}) \end{bmatrix} \right) \tag{24}$$

Furthermore, a solution to (21) is given by

$$\begin{bmatrix} \mathcal{Q}_{11} \\ \mathcal{Q}_{21} \end{bmatrix}_{opt} = \left[\begin{array}{ccc|c} A + BK & 0 & 0 & -\hat{L} \\ 0 & A & 0 & -\tilde{L} \\ 0 & B_Q C_2 & A_Q & (B_Q V_{21} + \hat{Z}_3 C_{11}^\mathsf{T})V_{11}^{-1} \\ \hline K & 0 & -E_2 C_Q & 0 \end{array}\right] \times \left[\begin{array}{c|c} A_{11} + \tilde{L}_1 C_{11} & \tilde{L}_1 \\ \hline C_{11} & I \end{array}\right] \tag{25}$$

where we have defined $\hat{L} = \tilde{L} - E_2\Psi C_{11}^\mathsf{T}V_{11}^{-1}$.

Proof. The proof is omitted, as it is analogous to that of Lemma 4. ∎

Remark 6. *If we isolate the optimal $\mathcal{Q}_{22}$ from Lemma 4, it simplifies greatly. Indeed, if we multiply (23) on the right by E_2, we obtain*

$$\mathcal{Q}_{22} = \left[\begin{array}{cc|c} A_{22} + B_{22}\tilde{K}_2 & B_{22}\hat{K} & 0 \\ 0 & A + LC & -LE_2 \\ \hline \tilde{K}_2 & \hat{K} & 0 \end{array}\right] \tag{26}$$

Similarly, the optimal $\mathcal{Q}_{11}$ from Lemma 5 simplifies to

$$\mathcal{Q}_{11} = \left[\begin{array}{cc|c} A + BK & \hat{L}C_{11} & -\hat{L} \\ 0 & A_{11} + \tilde{L}_1 C_{11} & -\tilde{L}_1 \\ \hline E_1^\mathsf{T}K & 0 & 0 \end{array}\right] \tag{27}$$

Remark 6 is the key observation that allows us to find a relatively simple analytic formula for the optimal controller. By substituting the result of Lemma 5 into Lemma 4, or vice-versa, we can obtain a simple set of equations that characterize the optimal controller.

We are now ready to prove the main result of the paper.

Proof of Theorem 1. Solving (8) is equivalent to simultaneously solving (20) and (21). To see why, write the optimality conditions for each one

$$E_2^\mathsf{T}P_{12}^\sim \left(\mathcal{P}_{11} + \mathcal{P}_{12} \begin{bmatrix} \mathcal{Q}_{11} & 0 \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \mathcal{P}_{21} \right) \mathcal{P}_{21}^\sim \in \begin{bmatrix} \mathcal{H}_2^\perp & \mathcal{H}_2^\perp \end{bmatrix}$$
$$P_{12}^\sim \left(\mathcal{P}_{11} + \mathcal{P}_{12} \begin{bmatrix} \mathcal{Q}_{11} & 0 \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \mathcal{P}_{21} \right) \mathcal{P}_{21}^\sim E_1 \in \begin{bmatrix} \mathcal{H}_2^\perp \\ \mathcal{H}_2^\perp \end{bmatrix}$$

and note that they are equivalent to

$$P_{12}^\sim \left(\mathcal{P}_{11} + \mathcal{P}_{12} \begin{bmatrix} \mathcal{Q}_{11} & 0 \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \mathcal{P}_{21} \right) \mathcal{P}_{21}^\sim \in \begin{bmatrix} \mathcal{H}_2^\perp & \mathcal{L}_2 \\ \mathcal{H}_2^\perp & \mathcal{H}_2^\perp \end{bmatrix} \tag{28}$$

which is the optimality condition for (8). It was shown in [5] that there always exists an optimal rational controller that solves 8. Therefore, there must also exist a simultaneous solution to (20) and (21).

Assumptions A1–A3 guarantee the existence of stabilizing solutions to (9)–(12). This is a standard result regarding CAREs. See for example [14, §13]. So we may apply Lemmas 4 and 5. Thus, there must exist Φ, Ψ, $\tilde{Z}_3$, and $\hat{Z}_3$ that simultaneously satisfy (22) and (24). Substituting (27) as (A_P, B_P, C_P) in (22) and similarly (26) as (A_Q, B_Q, C_Q) in (24), we obtain two augmented Sylvester equations. Algebraic manipulation shows that we must have

$$\tilde{Z}_3 = \begin{bmatrix} E_2^\mathsf{T}(\tilde{X} - X) & \Phi \end{bmatrix} \quad \text{and} \quad \hat{Z}_3 = \begin{bmatrix} \Psi \\ (\tilde{Y} - Y)E_1 \end{bmatrix}$$

where Φ and Ψ satisfy (13). This establishes existence and uniqueness of a solution to (13). Upon substituting these values back into (23) or (25), we obtain an explicit formula for the blocks of $\mathcal{Q}$. Upon simplification, we obtain (15).

Proof of Theorem 2 Obtain $\mathcal{Q}_{\mathrm{opt}}$ from Theorem 1, and transform using $\mathcal{K}_{\mathrm{opt}} = \mathcal{Q}_{\mathrm{opt}}(I + \mathcal{P}_{22}\mathcal{Q}_{\mathrm{opt}})^{-1}$. After some algebraic manipulations and reductions, we arrive at (16).

7 Conclusion

In Theorem 2, we give an explicit state-space formula for the solution to the two-player optimal control problem with output feedback. The construction requires solving four standard algebraic Riccati equations, as well as a pair of coupled linear equations. The optimal controller, which was not previously known, has twice as many states as the original system.

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