



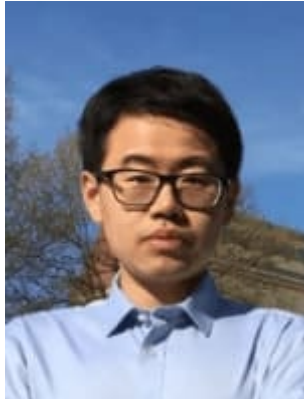
HARVARD

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The Role of Prediction in Online Optimal Control

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Electrical Engineering and Applied Mathematics

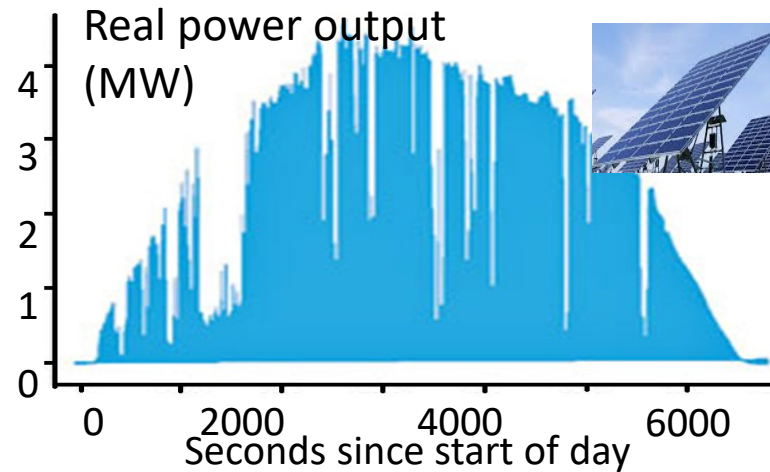
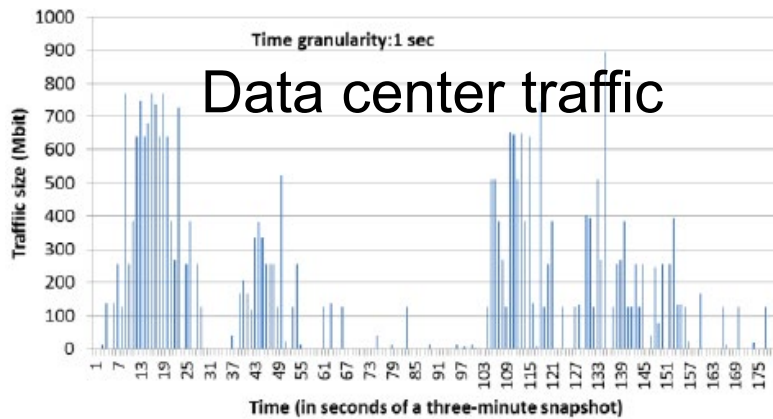
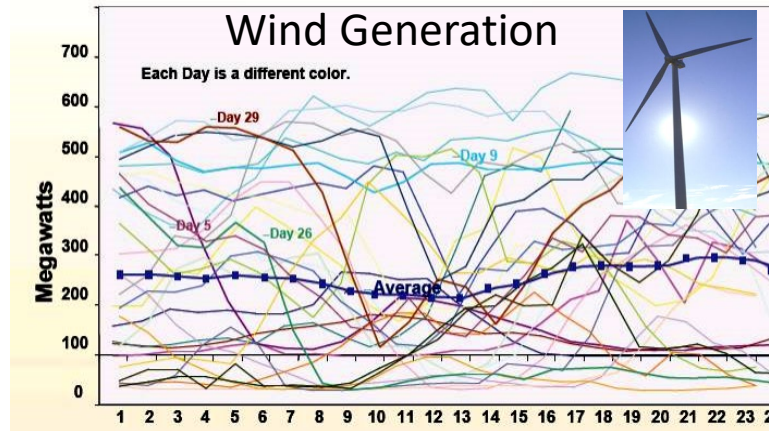


Acknowledgement:

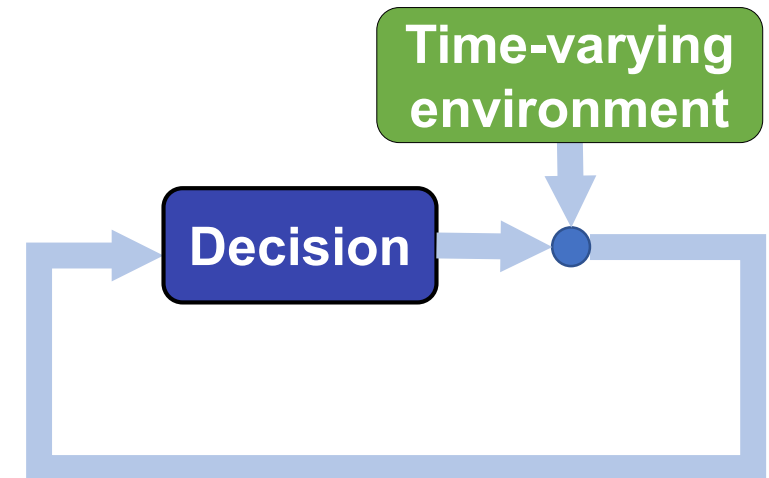
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ACC workshop, July 09, 2019

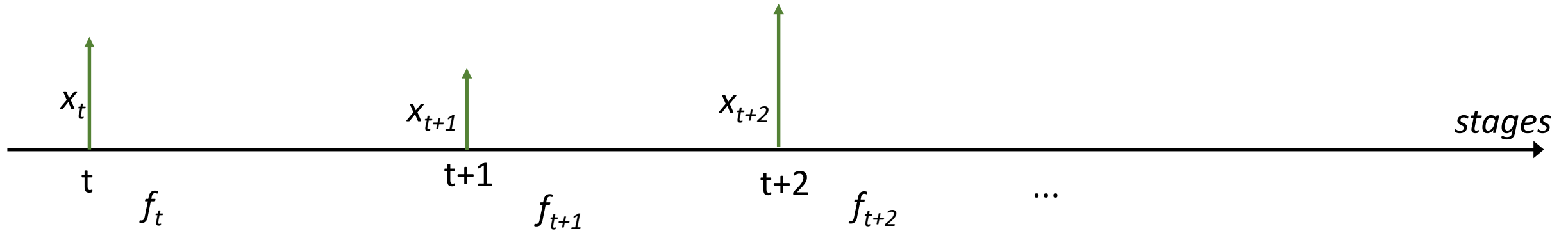
Uncertainties in decision-making problems are ubiquitous



Online decision-making under a time-varying environment



Online optimization



Input: A convex set $X \subset \mathbb{R}^n$

for: $t = 1, 2, \dots$

make a decision $x_t \in X$

receive a convex cost function $f_t : X \rightarrow \mathbb{R}$

suffer cost $f_t(x_t)$

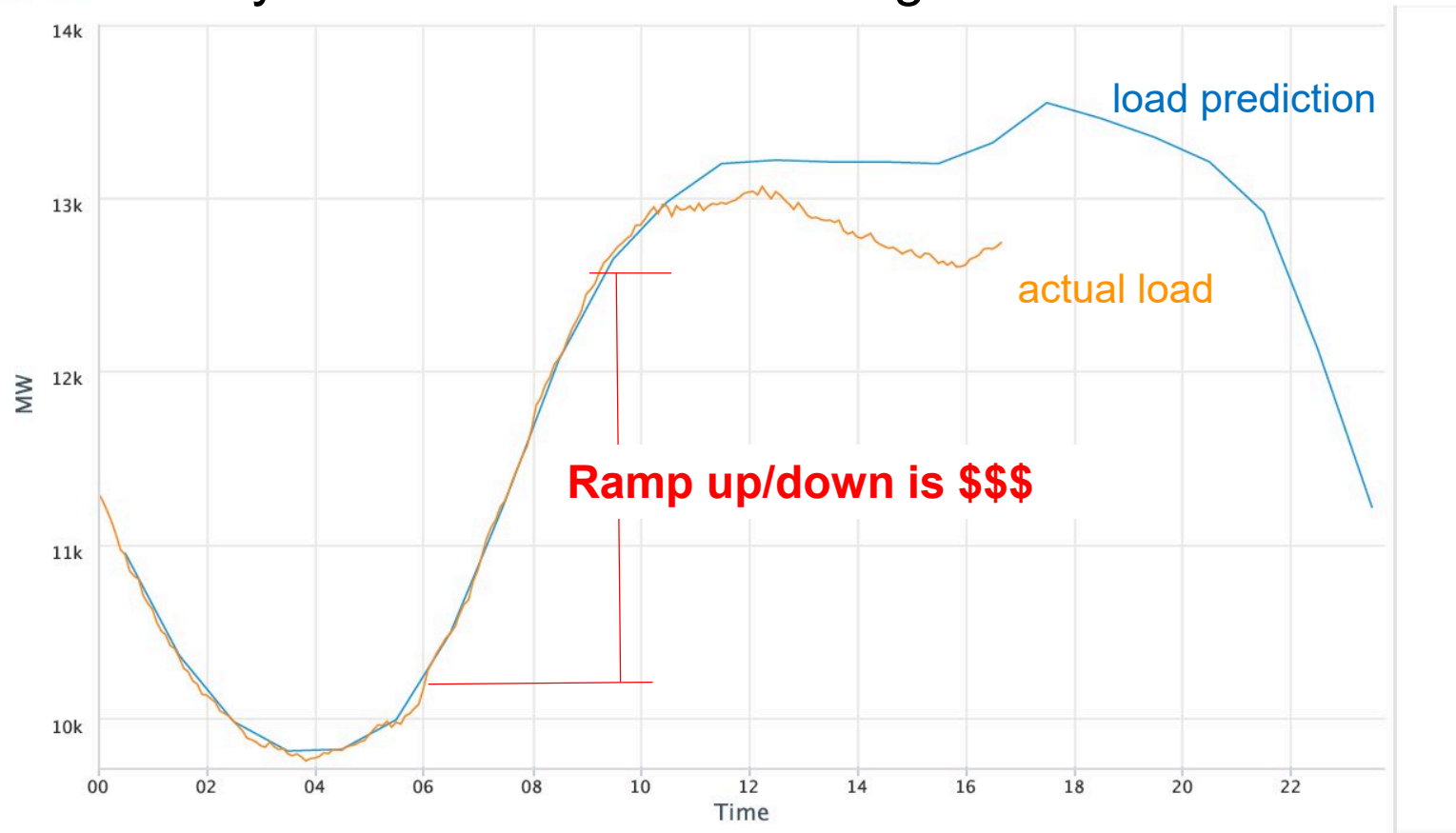
End

Goal: minimize total costs by making decisions without knowing the future

System dynamics: Switching/control costs

6/23/2018

System Load from New England ISO



Updated: 06/23/2018 04:42 PM

Economic Dispatch:
ramping cost

Source: New England ISO

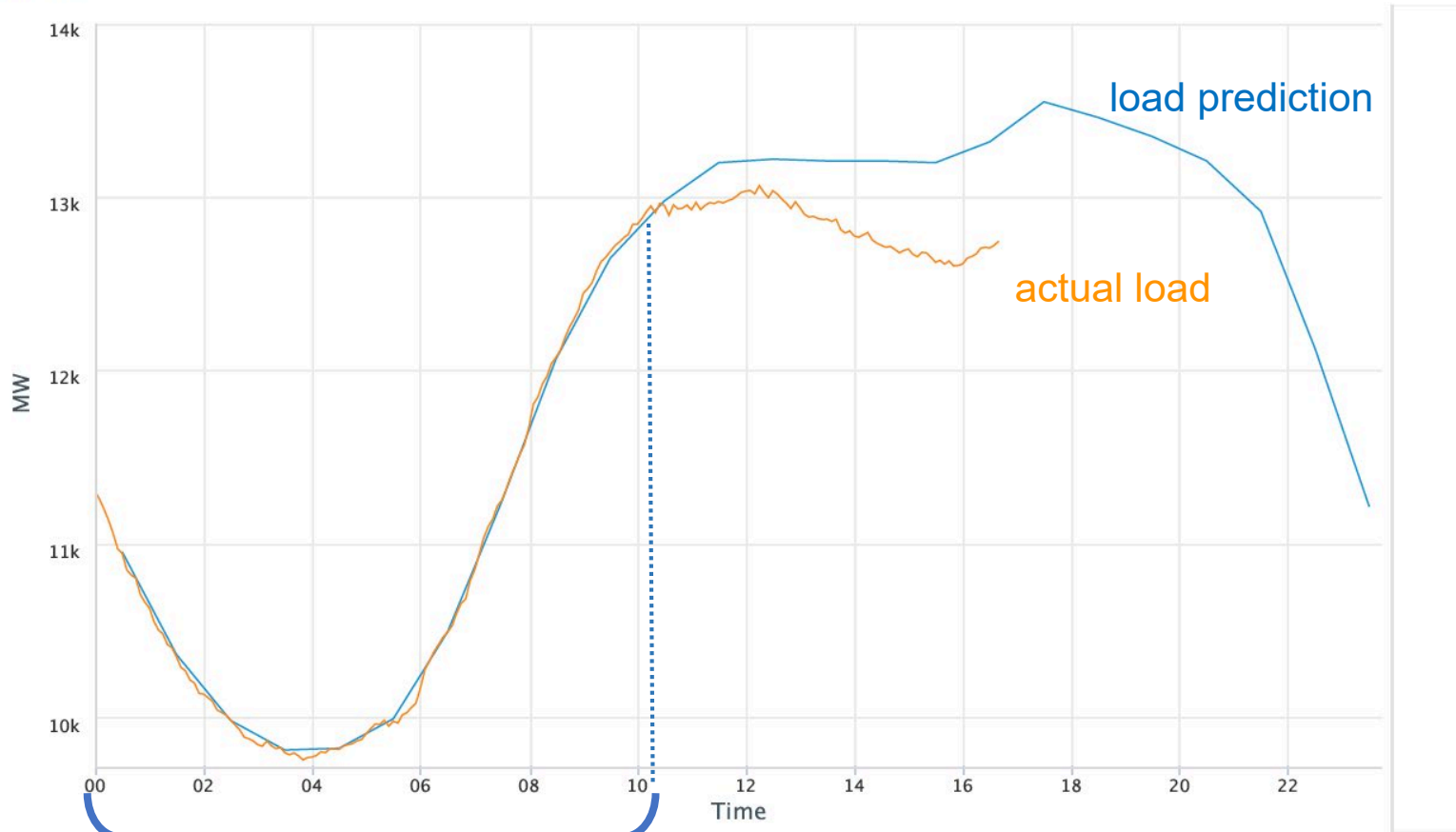


Vehicle racking:
energy cost

Short-term prediction is possible

System Load from New England ISO

6/23/2018



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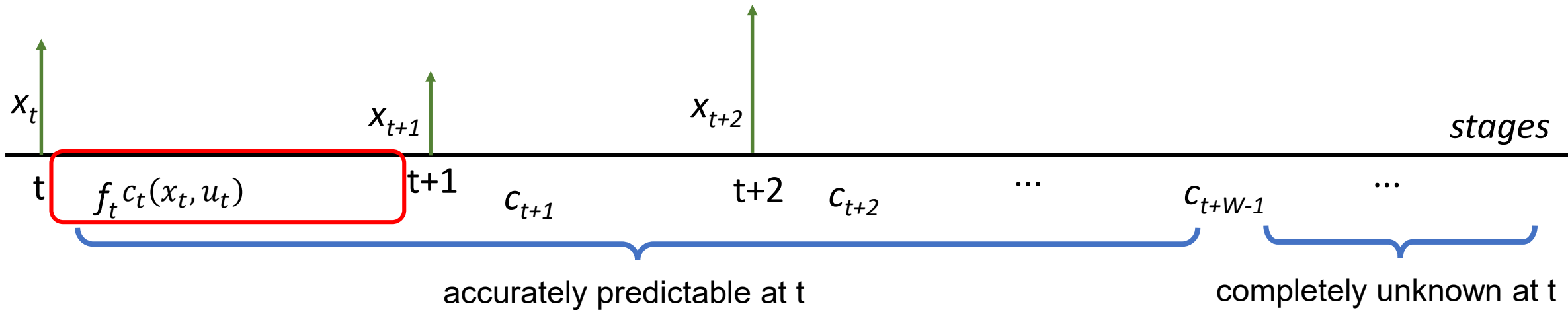
Source: New England ISO

Good short term prediction



Vehicle racking

Online control with linear system dynamics



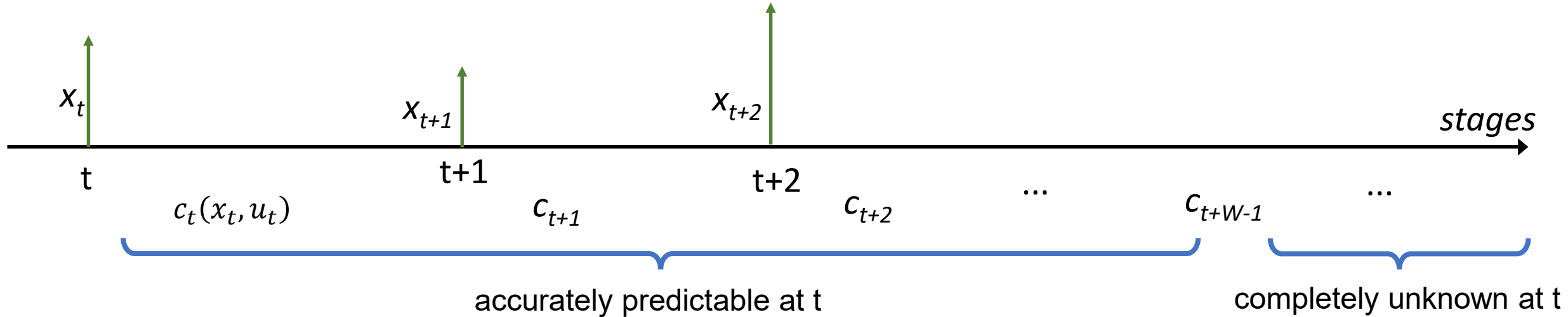
- System follows dynamics $x_t = g(x_{t-1}, u_t) \rightarrow x_t = Ax_{t-1} + Bu_t$
- Each time step $t = 1, \dots, T$ is involved with a stage cost $c_t(x_t, u_t) = f_t(x_t) + g_t(u_t)$
Example: LQR: $\frac{1}{2}(x_t - \theta_t)^T Q_t (x_t - \theta_t) + u_t^T R_t u_t$
- At time t , know accurate W -step prediction c_t, \dots, c_{t+W-1} , $W \geq 0$

Online algorithm \mathcal{A} :

Pick action $u_t^{\mathcal{A}} \in U$ using prediction and history to minimize the total cost $C = \sum_t^T c_t$.

- Prior information: A, B, x_0 are known

Online control with linear system dynamics



Performance metric: **dynamic** regret (competitive difference)

$$\text{Regret} := J(\mathbf{x}, \mathbf{u}) - \min_{\mathbf{x}, \mathbf{u}} J(\mathbf{x}, \mathbf{u})$$

Online cost

Optimal offline cost

where $J(\mathbf{x}, \mathbf{u})$ denotes the total cost function $J(\mathbf{x}, \mathbf{u}) = \sum_{t=1}^T c_t(x_t, u_t)$.

Literature review

MPC and economic MPC

[Mayne et al. 1999] [Morari et al. 1999] [Rawlings et al. 2012]...

- **Transient optimality** for **time-invariant** economic MPC
[Grüne et al 2014] [Angeli et al 2012]...
- **Asymptotic** analysis for economic MPC with **time-varying costs**
[Angeli et al 2015] [Ellis et al 2013] [Ferramosca, et al. 2010] [Ferramosca et al 2014]
[Alessandretti et al 2016] [Angeli et al 2016]

Nonasymptotic analysis for time-varying costs?

What is achievable and what is not?—Role of prediction

Computational efficient way to achieve the performance limit?

Literature review

Online convex optimization with switching costs

[Lin et al. 2012] [Lin et al. 2013] [Chen et al. 2015] [Chen et al. 2016] [Li et al 2018] ...

- A **special** case of control problems: $c_t(x_t, u_t) = f(x_t) + \frac{\beta}{2} \|u_t\|^2$
$$x_{t+1} = x_t + u_t$$
- Tools are developed for **nonasymptotic** regret analysis

Can we borrow ideas from online optimization to design control algorithms?

Can we borrow techniques to prove nonasymptotic bounds?

Brief introduction of online optimization with switching cost

Special case of online control

$$\min_{x_t, u_t} \sum_t (f_t(x_t) + \frac{\beta}{2} \|u_t\|^2)$$

$$\text{s.t. } x_{t+1} = x_t + u_t$$



Online opt with switching costs

$$\min_{x_t} \sum_t (f_t(x_t) + \frac{\beta}{2} \|x_{t+1} - x_t\|^2)$$

Main Messages:

Given any online optimization algorithm **without** using prediction, **additional W steps of gradient calculation** can further reduce the regret by $O(r^W)$.

Offline gradient descent

Offline optimization:

$$\min_{x \in X \times \dots \times X} C_1^T(x) \quad \text{where } C_1^T(x) := \sum_{t=1}^T \left(f_t(x_t) + \frac{\beta}{2} \|x_t - x_{t-1}\|^2 \right)$$

Offline gradient descent:

$$\frac{\partial C_1^T}{\partial x_t}(x^{(k-1)})$$

$$\overset{\text{\#iter}}{x_t^{(k)}} = \Pi_X \left[x_t^{(k-1)} - \eta \left(\nabla f_t(x_t^{(k-1)}) + \beta(x_t^{(k-1)} - x_{t-1}^{(k-1)}) - \beta(x_{t+1}^{(k-1)} - x_t^{(k-1)}) \right) \right]$$

stage $t=1 \dots T$

$k=0$ (initialization)

$$\dots, x_{t+W-2}^{(0)}, x_{t+W-1}^{(0)}, x_{t+W}^{(0)}, \dots$$

OCO without prediction

$k=1$

$$x_{t+W-1}^{(1)}$$

$$f_{t+W-1}$$

\vdots

\vdots

\vdots

$k=W-2$

$$x_t^{(W-2)}, x_{t+1}^{(W-2)}, x_{t+2}^{(W-2)},$$

$$f_{t+1}$$

$k=W-1$

$$x_{t-1}^{(W-1)}, x_t^{(W-1)}, x_{t+1}^{(W-1)},$$

$$f_t$$

$k=W$

$$x_t^{(W)}$$

Key observation:

Predictable at t

To compute $x_t^{(W)}$, need $f_t \dots f_{t+W-1}$.

Initialization: any OCO without prediction

RHGD

Offline gradient descent:

$$x_t^{(k)} = \Pi_X \left[x_t^{(k-1)} - \eta \left(\nabla f_t(x_t^{(k-1)}) + \beta(x_t^{(k-1)} - x_{t-1}^{(k-1)}) - \beta(x_{t+1}^{(k-1)} - x_t^{(k-1)}) \right) \right]$$

Additional Computation per stage:

- $(W + 1)$ projected gradient evaluations

RHGD:

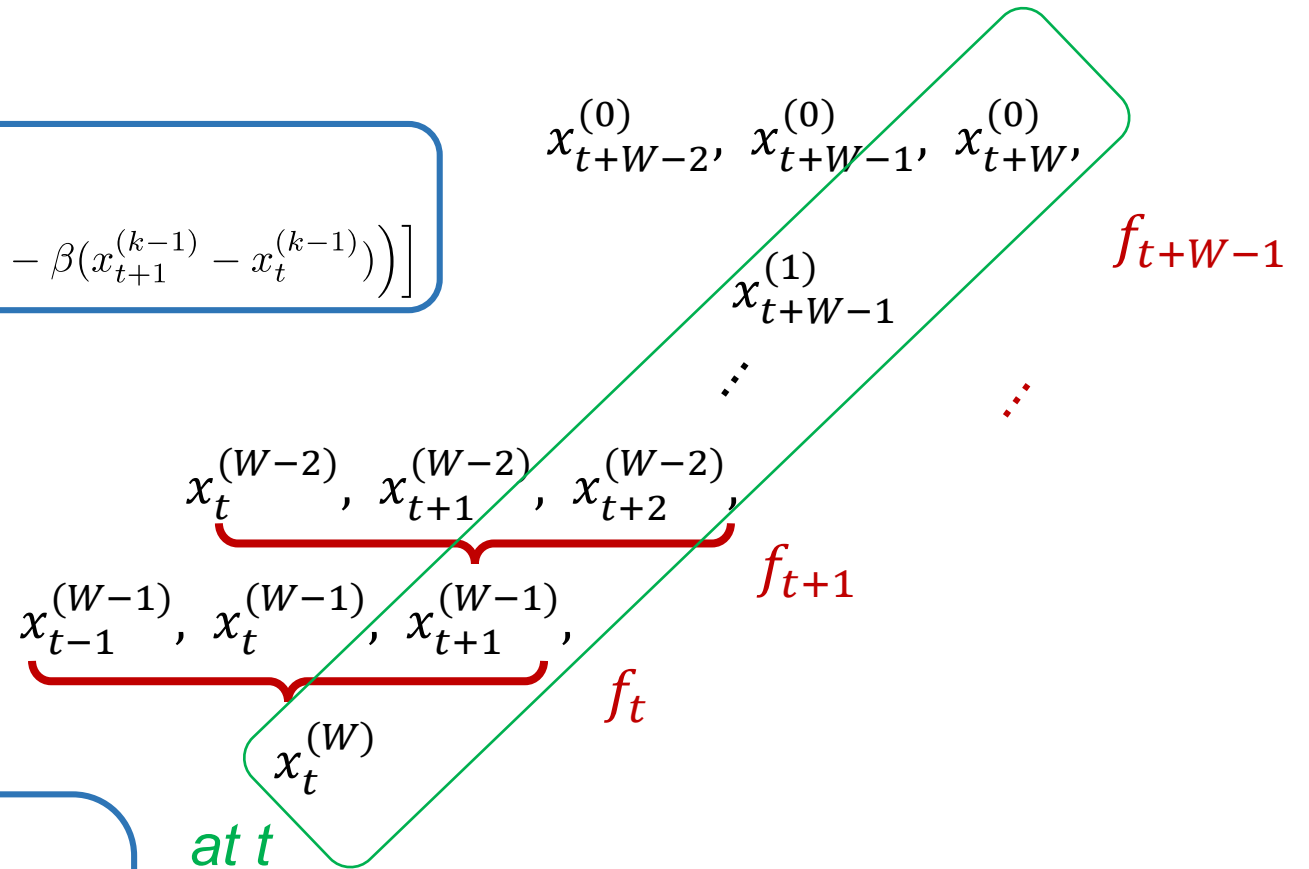
For $t = 1, 2 \dots$

Receive prediction $f_t \dots f_{t+W-1}$

Initialize $x_{t+W}^{(0)}$,

Backward compute $x_{t+W-1}^{(1)} \dots, x_t^{(W)}$

Output $x_t^{(W)}$



Key observation:

Predictable at t

To compute $x_t^{(W)}$, need $f_t \dots f_{t+W-1}$.

Initialization: any OCO without prediction

RHAG: Nesterov's accelerated gradient

RHGD:

Prediction f_t, \dots, f_{t+W-1} allows W iterations of offline Nesterov's accelerated gradient

Offline optimization: $\min_{x \in X \times \dots \times X} C_1^T(x)$ where $C_1^T(x) := \sum_{t=1}^T \left(f_t(x_t) + \frac{\beta}{2} \|x_t - x_{t-1}\|^2 \right)$

Offline Nesterov's accelerated gradient :

$$x_t^{(k)} = \Pi_X \left[y_t^{(k-1)} - \eta \left(\nabla f_t(y_t^{(k-1)}) + \beta(y_t^{(k-1)} - y_{t-1}^{(k-1)}) - \beta(y_{t+1}^{(k-1)} - y_t^{(k-1)}) \right) \right]$$

$$y_t^{(k)} = (1 + \lambda)x_t^{(k)} - \lambda x_t^{(k-1)}$$

k=W-1

$$\left(x_{t-1}^{(W-1)}, y_{t-1}^{(W-1)} \right), \left(x_t^{(W-1)}, y_t^{(W-1)} \right), \left(x_{t+1}^{(W-1)}, y_{t+1}^{(W-1)} \right),$$

k=W

$$\left(x_t^{(W)}, y_t^{(W)} \right)$$

RHAG: Nesterov's accelerated gradient

RHGD:

Prediction f_t, \dots, f_{t+W-1} allows W iterations of offline Nesterov's accelerated gradient

Offline optimization: $\min_{x \in X \times \dots \times X} C_1^T(x)$ where $C_1^T(x) := \sum_{t=1}^T \left(f_t(x_t) + \frac{\beta}{2} \|x_t - x_{t-1}\|^2 \right)$

Offline Nesterov's accelerated gradient :

$$x_t^{(k)} = \Pi_X \left[y_t^{(k-1)} - \eta \left(\nabla f_t(y_t^{(k-1)}) + \beta(y_t^{(k-1)} - y_{t-1}^{(k-1)}) - \beta(y_{t+1}^{(k-1)} - y_t^{(k-1)}) \right) \right]$$

$$y_t^{(k)} = (1 + \lambda)x_t^{(k)} - \lambda x_t^{(k-1)}$$

*Similar for other fast gradient methods, such as **Triple Momentum (TM)** method (Scoy, Freeman, Lych, 2018)*

Regret bounds: exponential decay with prediction length W

Theorem 1. (informal)

For any initialization method ψ , under proper stepsizes

$$\text{Regret}(RHGD) \leq O \left(\left(\frac{Q_f - 1}{Q_f} \right)^W \text{Regret}(\psi) \right)$$

$$\text{Regret}(RHAG) \leq O \left(\left(\frac{\sqrt{Q_f - 1}}{\sqrt{Q_f}} \right)^W \text{Regret}(\psi) \right)$$

$$\text{Regret}(RHTM) \leq O \left(\left(\frac{\sqrt{Q_f - 1}}{\sqrt{Q_f}} \right)^{2W} \text{Regret}(\psi) \right)$$

Exp. decay with W

where Q_f is the condition number of the total cost.

General cases?

Simple Case of Online Control

$$\min_{x_t, u_t} \sum_t (f_t(x_t) + \frac{\beta}{2} \|u_t\|^2)$$

$$\text{s.t. } x_{t+1} = x_t + u_t$$



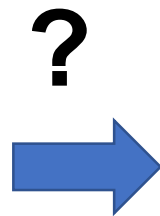
Online Opt with Switching Costs

$$\min_{x_t} \sum_t (f_t(x_t) + \frac{\beta}{2} \|x_{t+1} - x_t\|^2)$$

General linear dynamics

$$\min_{x_t, u_t} \sum_t (f_t(x_t) + g_t(u_t))$$

$$\text{s.t. } x_{t+1} = Ax_t + Bu_t$$



Finite-coupling costs have finite-coupling gradient

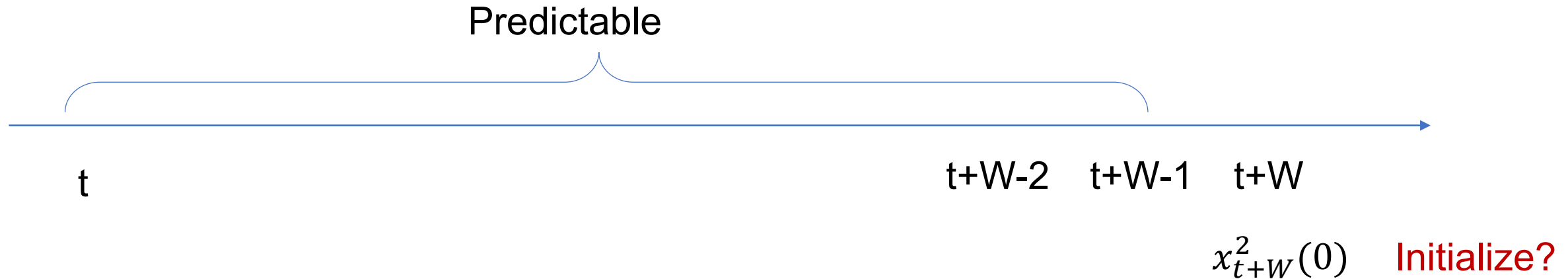
Finite-coupling costs: $C(\mathbf{x}^2) := \sum_t \tilde{c}_t(x_{t-1}^2, x_t^2, x_{t+1}^2)$

Finite-coupling partial derivative:

$$\frac{\partial C}{\partial x_t^2}(\mathbf{x}^2) = \frac{\partial(\tilde{c}_{t-1} + \dots + \tilde{c}_{t+1})}{\partial x_t^2}(x_{t-2}^2(k), \dots, x_{t+2}^2(k))$$

RHGD, RHAG, RHTM can be extended to the general cases

An initialization method: follow optimal steady states



Initialization algorithm:

- Solve steady state optimization at $t+W-1$:

$$(\bar{x}_{t+W-1}, \bar{u}_{t+W-1}) = \arg \min_{x=Ax+Bu} c_{t+W-1}(x, u)$$

- Initialize by *following the optimal steady state*:

$$x_{t+W}^2(0) = \bar{x}_{t+W-1}$$

Regret bound of the initialization method

Theorem 2. (informal)

For the online LQR problem, applying the optimal steady state at each time,

$$\text{Regret}(\psi) \leq O \left(\sum_t (\|P_t^* - P_{t-1}^*\|^2 + \|\bar{x}_t^* - \bar{x}_{t-1}^*\|^2 + \|\beta_t^* - \beta_{t-1}^*\|^2) \right)$$

where P_t^* is the solution to the algebraic Riccati equation defined by Q_t, R_t ; β_t^* is the drifting term in the optimal controller $u = K_t^* x + K_t' \beta_t^*$ to the LQR problem defined by Q_t, R_t, θ_t .

variation of costs:
path length

Small variation of costs



Easy.

Future is similar; thus well-estimated by history

Large variation of costs



Difficult!

Other initial. alg: online control without prediction [Abbasi-Yadkori et al 2014], [Goel et al. 2019]; online gradient [Li et al 2019]

Regret bound of RHTM with opt-steady-state initialization

Corollary 1. (informal)

For the online LQR problem, applying RHTM with the optimal steady state as initialization,

$$\text{Regret}(RHTM) \leq O \left(\left(\frac{\sqrt{Q_f} - 1}{\sqrt{Q_f}} \right)^{2K} \sum_t (\|P_t^* - P_{t-1}^*\|^2 + \|\bar{x}_t^* - \bar{x}_{t-1}^*\|^2 + \|\beta_t^* - \beta_{t-1}^*\|^2) \right)$$

where P_t^* is the solution to the algebraic Riccati equation defined by Q_t, R_t ; β_t^* is the drifting term in the optimal controller $u = K_t^* x + K_t' \beta_t^*$ to the LQR problem defined by Q_t, R_t, θ_t .

Special case: Q and R not change

Theorem. (informal)

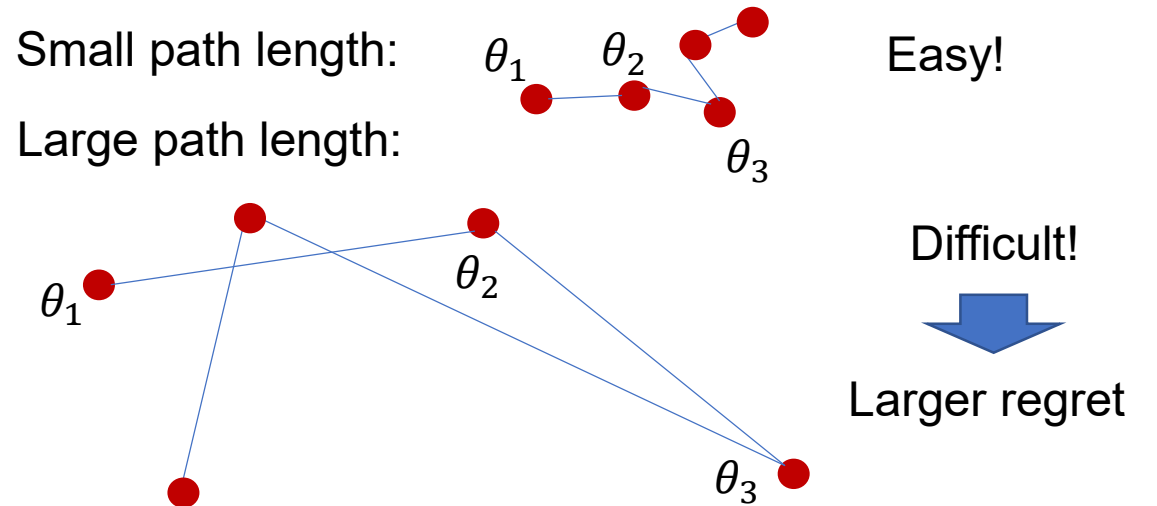
For LQR tracking where stage cost is $c_t(x_t, u_t) := \frac{1}{2}(x_t - \theta_t)^T Q(x_t - \theta_t) + u_t^T R u_t$,

$$C(\text{RHTM}) - C(\text{Offline Optimal}) \leq O \left(\left(\frac{\sqrt{Q_f} - 1}{\sqrt{Q_f}} \right)^{2K} \sum_t \|\theta_t - \theta_{t-1}\|_2 \right)$$

variation of costs
path length

where $K = \lfloor \frac{W}{p} \rfloor$, p is maximum block size, Q_f is the conditional number.

Tracking cost: $\|x_t - \theta_t\|^2$
 tracker's location at t target's location at t



Dependence on W is nearly optimal

Theorem. (informal)

For LQR tracking where stage cost is $c_t(x_t, u_t) := \frac{1}{2}(x_t - \theta_t)^T Q(x_t - \theta_t) + u_t^T R u_t$,

$$C(\text{RHTM}) - C(\text{Offline Optimal}) \leq O \left(\left(\frac{\sqrt{Q_f} - 1}{\sqrt{Q_f}} \right)^{2K} \sum_t \|\theta_t - \theta_{t-1}\|_2 \right)$$

where $K = \lfloor \frac{W}{p} \rfloor$, p is maximum block size, Q_f is the conditional number.

**nearly
optimal**

Theorem 3 [Fundamental lower bound] (informal)

For any online algorithm \mathcal{A} , given any $1 < W < T$ and $p \geq 1$, there exists $A, B, \{\theta_t\}_{t=1}^T$ s.t.

$$C(\mathcal{A}) - C(\text{Offline Optimal}) \geq \Omega \left(\left(\frac{\sqrt{Q_f} - 1}{\sqrt{Q_f} + 1} \right)^{2K} \sum_t \|\theta_t - \theta_{t-1}\|_2 \right)$$

where $K = \lfloor \frac{W}{p} \rfloor$, p is maximum block size, Q_f is the conditional number.

Proof sketch for the special case.

Special case of online control

$$\min_{x_t, u_t} \sum_t (f_t(x_t) + \frac{\beta}{2} \|u_t\|^2)$$

$$\text{s.t. } x_{t+1} = x_t + u_t$$



Online opt with switching costs

$$\min_{x_t} \sum_t (f_t(x_t) + \frac{\beta}{2} \|x_{t+1} - x_t\|^2)$$

Proof sketch: $\min_{x_t} \sum_t (f_t(x_t) + \frac{\beta}{2} \|x_{t+1} - x_t\|^2)$ & when path length is T .

- Construct $f_t(x_t, u_t) = \|x_t - \theta_t\|^2$, with random $\theta_1, \dots, \theta_n$ i.i.d. $E\theta_i = 0$, $\text{var}(\theta_i) = 1$, bounded. Path length = $O(T)$
- **Optimal offline action**

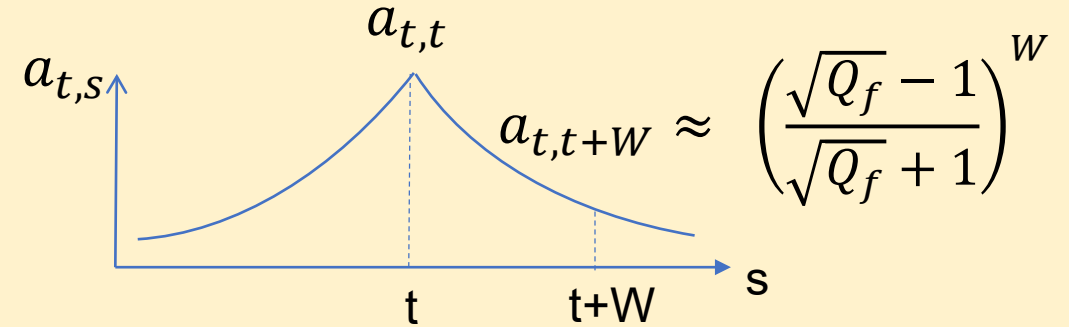
$$x_t^* = \underbrace{[a_{t,1}\theta_1 + \dots + a_{t,t}\theta_t + a_{t,t+1}\theta_{t+1} + \dots + a_{t,t+W-1}\theta_{t+W-1}]}_{x_t^t} + \underbrace{[a_{t,t+W}\theta_{t+W} + \dots + a_{t,T}\theta_T]}_{x_t^{T-t}}$$

- **Online action:** only know $\theta_1, \dots, \theta_{t+W-1}$,

Key 1: x_t^t is a function of $\theta_1, \dots, \theta_{t+W-1}$

Error: $E|x_t - x_t^*|^2 = E|x_t - x_t^t - x_t^{T-t}|^2$
 $= E|x_t - x_t^t|^2 + E|x_t^{T-t}|^2$
 $\geq a_{t,t+W}^2 + \dots + a_{t,T}^2$

Key 2: $a_{t,t+W}, \dots$ decays exponentially

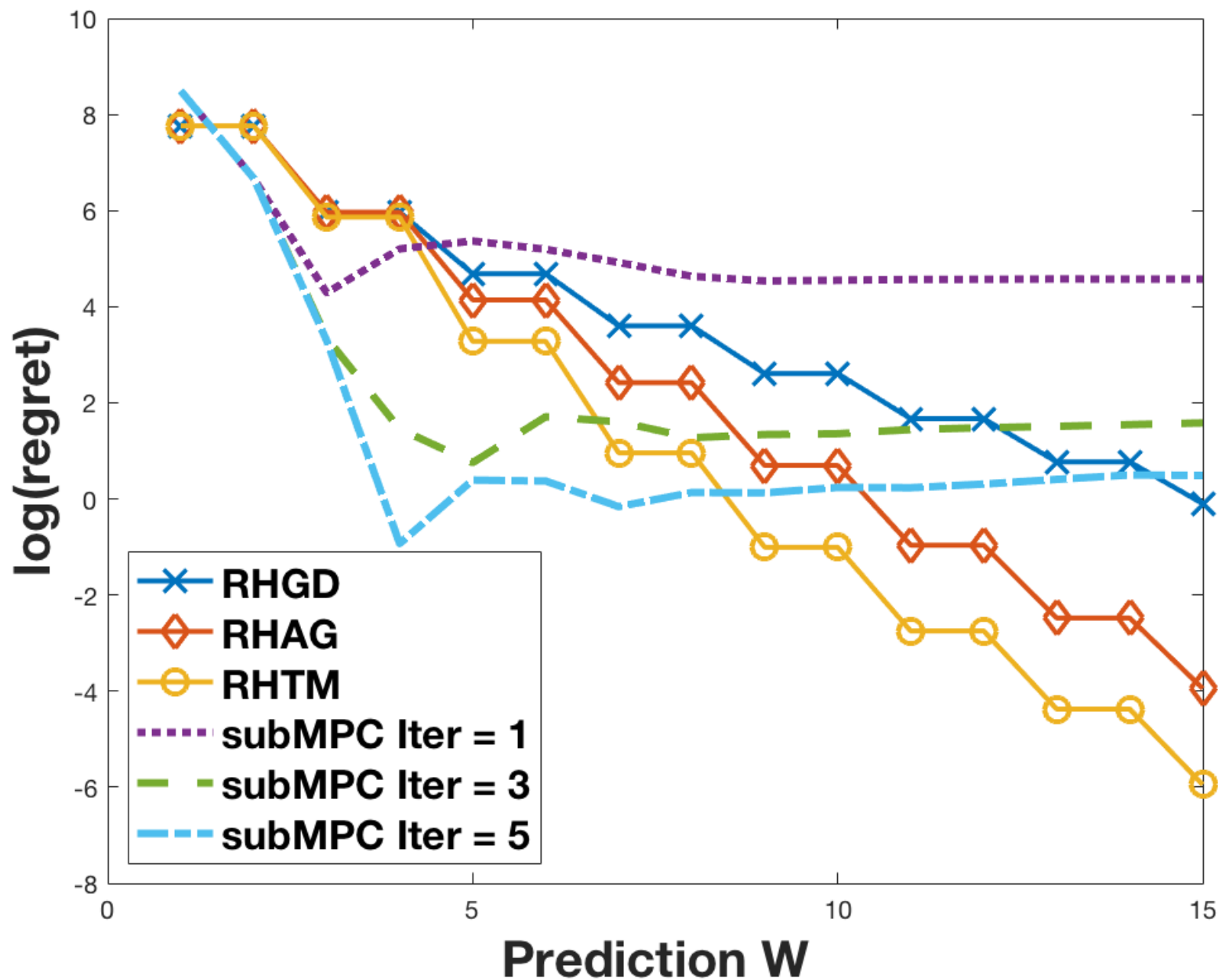


- **Regret:** By strong convexity,

$$E(C_1^T(x) - C_1^T(x^*)) \geq \alpha/2 \sum_{t=1}^T E\|x_t - x_t^*\|^2 \geq \frac{\alpha}{2} \left(\frac{\sqrt{Q_f} - 1}{\sqrt{Q_f} + 1} \right)^{2W} T \quad \text{Path length}$$

Numerical studies: Comparison with sub-MPC

- LTI system: two states
- Single input
- $p=2$



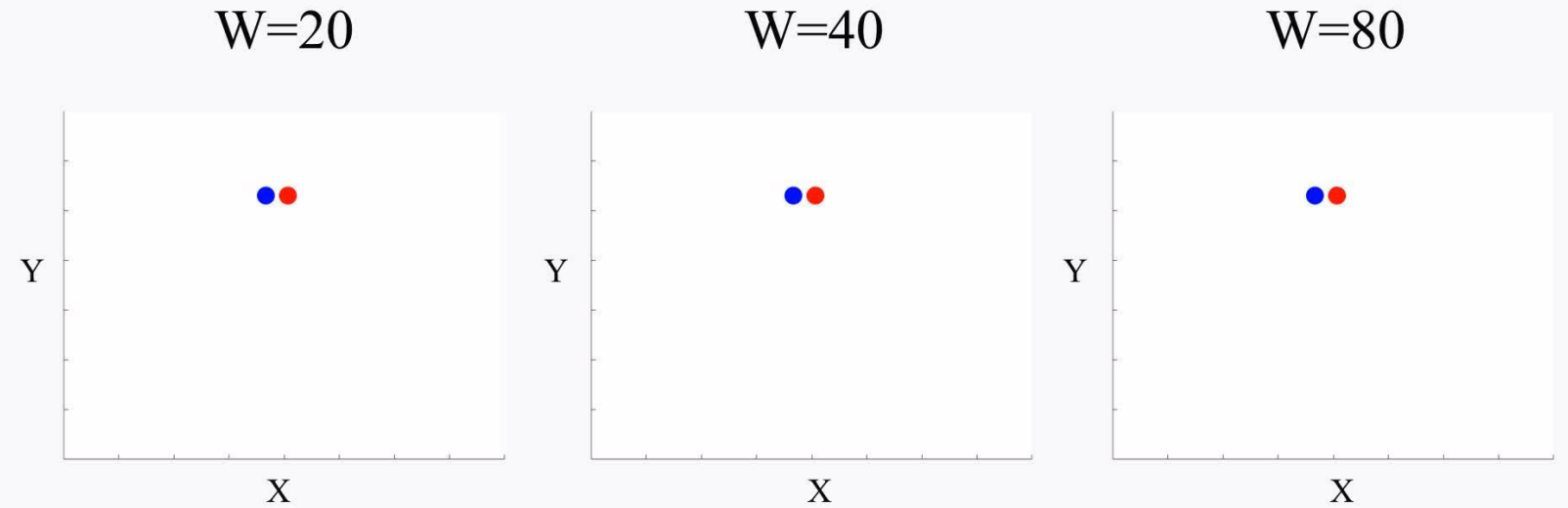
Numerical example: Two wheel robot dynamic tracking



$$\dot{x} = v \cos \delta$$

$$\dot{y} = v \sin \delta$$

$$\dot{\delta} = \omega$$



Two-wheel robot trajectory tracking

Summary

- Given any online optimal control algorithm without using prediction, additional W step of gradient calculations can further reduce the regret by $O(r^K)$, $K = \lfloor \frac{W}{p} \rfloor$
- Fundamental lower bound for any online algorithm: Dynamic regret **exponential** decays with $K = \lfloor \frac{W}{p} \rfloor$
- The dependence on W is nearly optimal

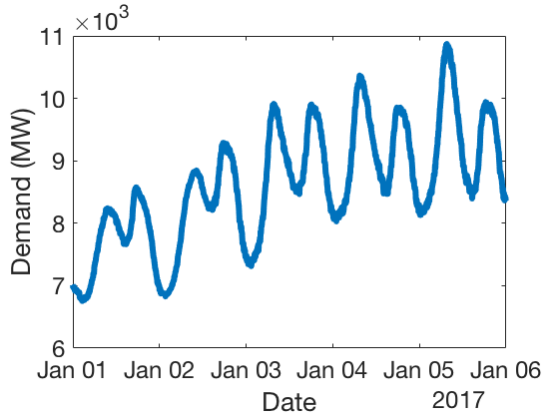
Thank you!

Ongoing and Future work:

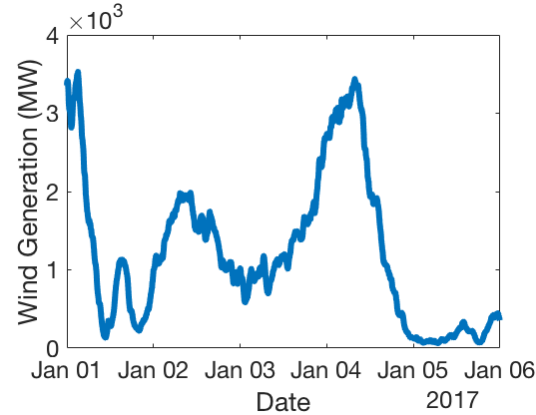
- Stochastic/Noisy prediction
- Study optimal control with constraints; the dynamic regret bounds of MPC
- Online control problem for general dynamical systems including Markov decision models
- Other meaningful metrics for online performance analysis

Simulation 1: economic dispatch

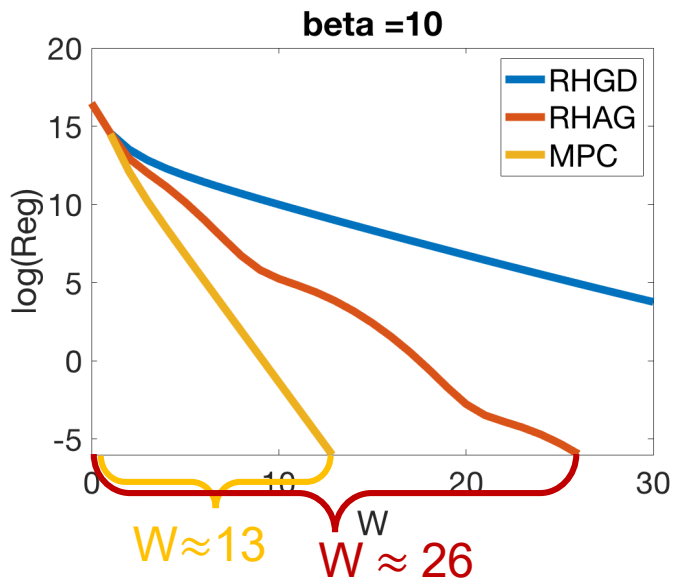
(a) Power demand



(b) Wind generation



(c) Dynamic regrets



(d) Running time /second

TABLE I
RUNNING TIME OF RHGD, RHAG AND MPC

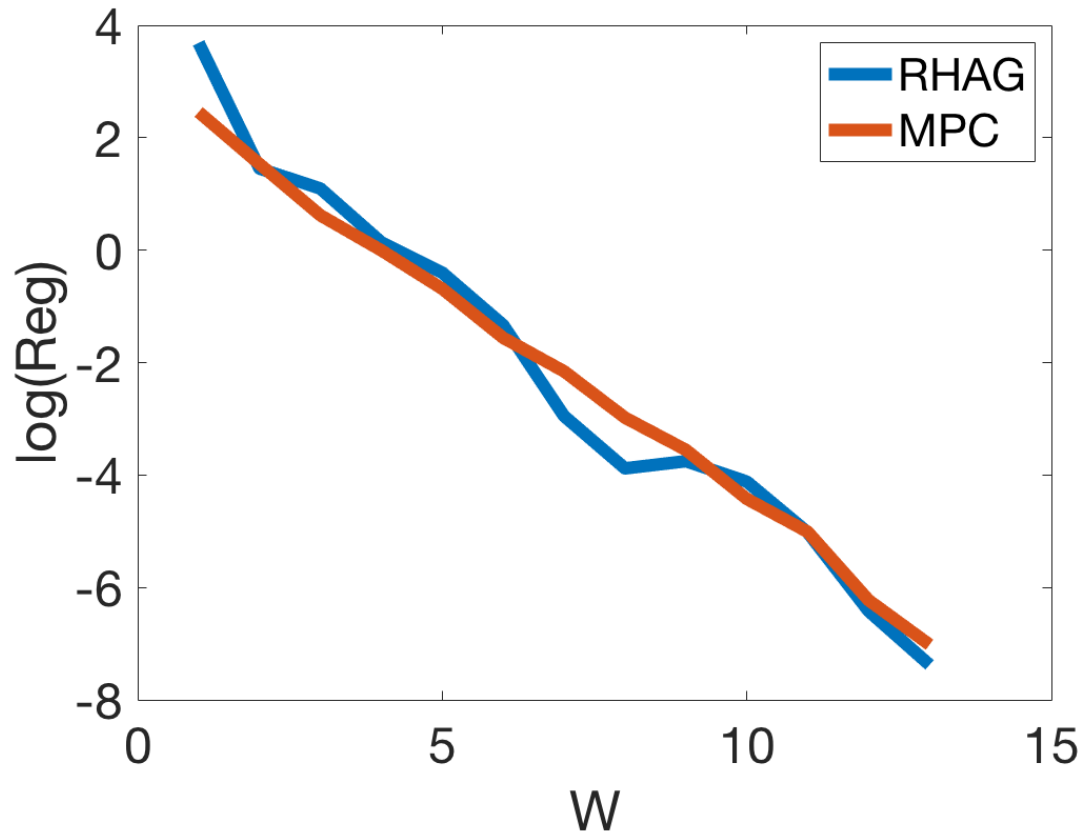
\mathcal{A} \ W	5	10
RHGD	8.8781×10^{-5}	1.4923×10^{-4}
RHAG	1.0416×10^{-4}	1.9052×10^{-4}
MPC	0.0105	0.0110

Setup:

- stage: 5 min
- horizon: 5 days ($T=1440$)
- data source: Bonneville Power Administration (BPA) controlled area
- $n=3$ generators
- $Q_f = 11.75$
- stage cost:
 - 1) cost of generation
 - 2) cost of uncleared demand
- Matlab (quadprog())

Simulation 2

Dynamic regrets

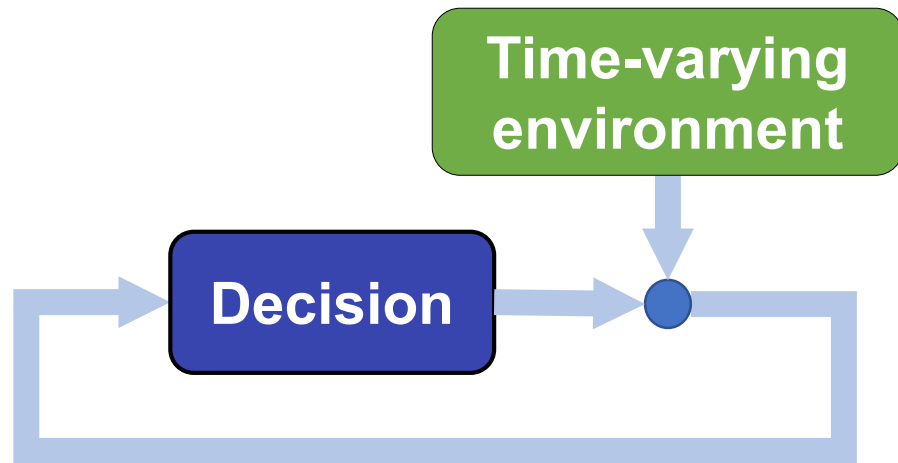


Setup:

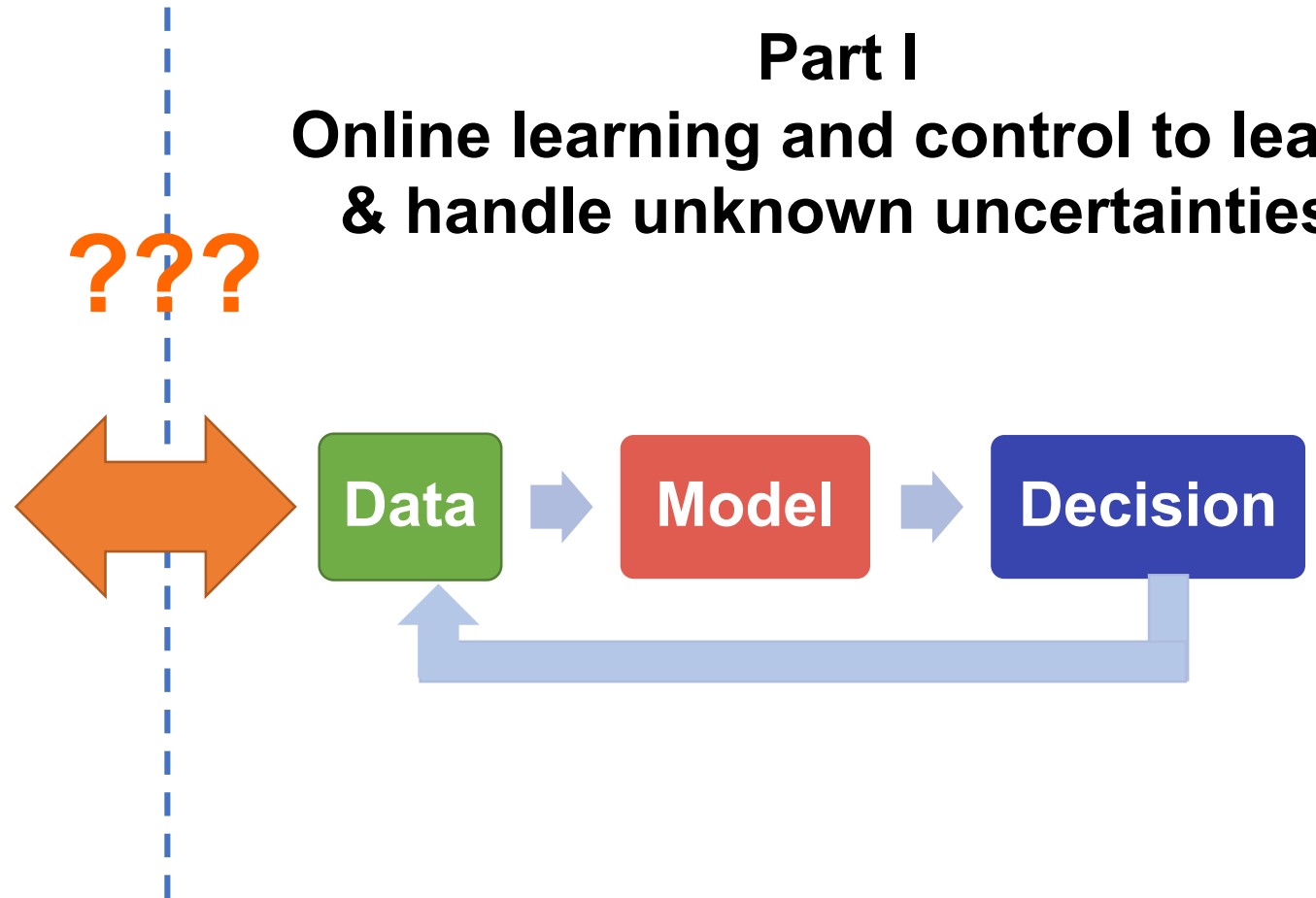
- horizon: $T=16$
- Scalar decision variable
- $Q_f \approx 47$
- stage cost:
$$\frac{1}{2}(x_t - \theta_t)^2$$
- θ_t picked from $\{0,4\}$

Summary: Online Learning and Decision Making

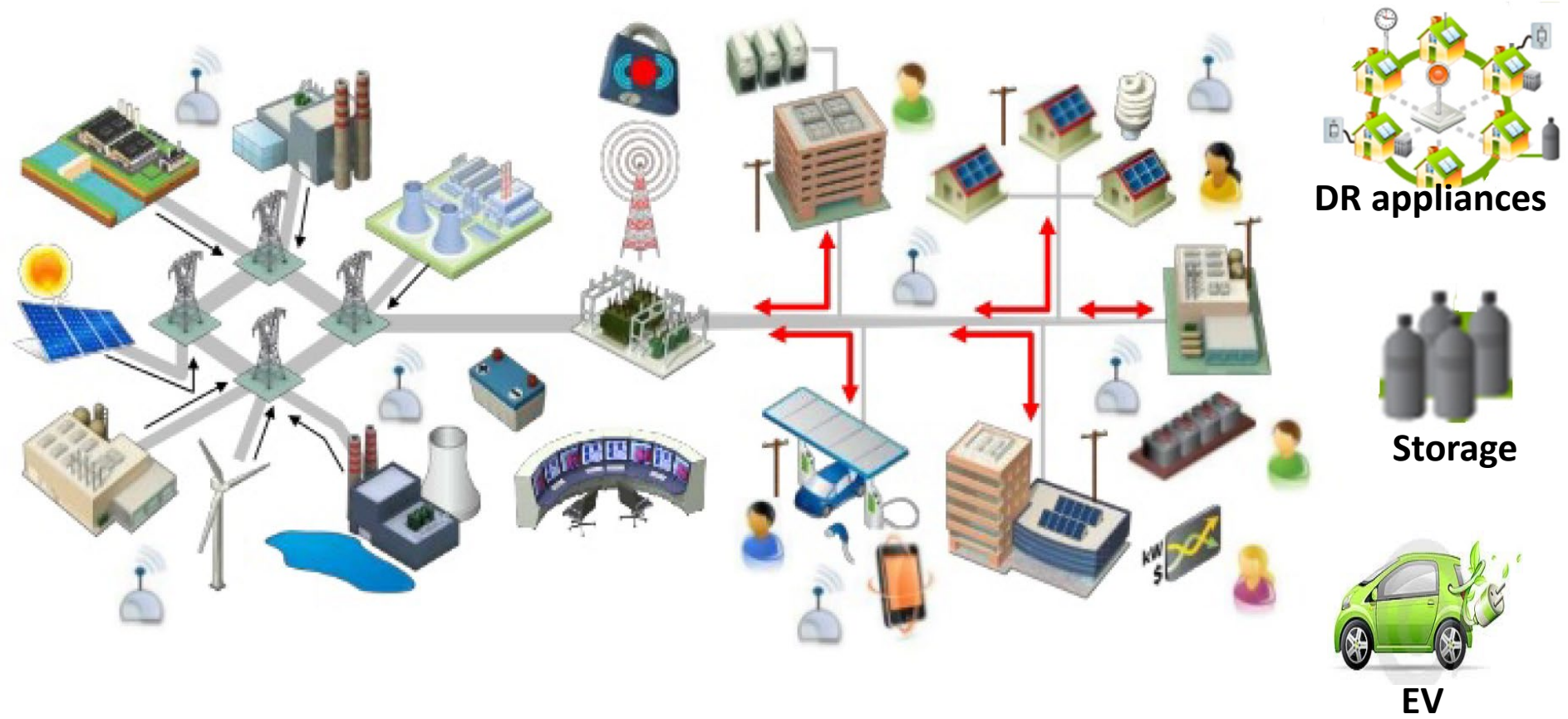
Part II
Online decision-making under
a time-varying environment



Part I
Online learning and control to learn
& handle unknown uncertainties

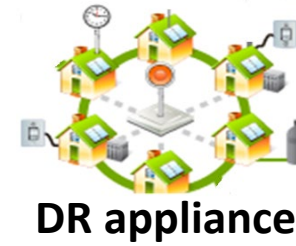
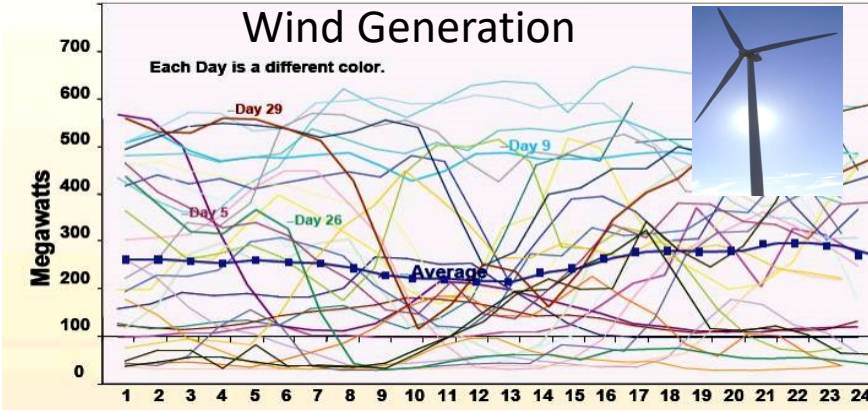


Thank you!

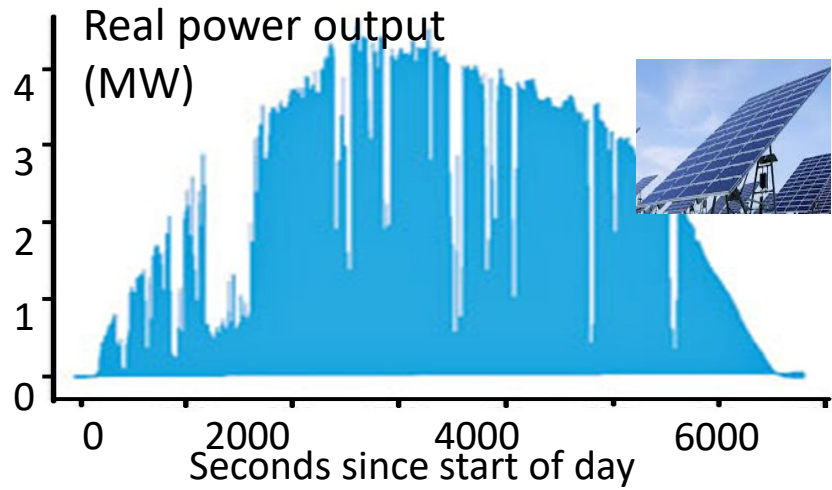


Supply = Demand

Increasing uncertainties in the grid



Responsive ← Unresponsive

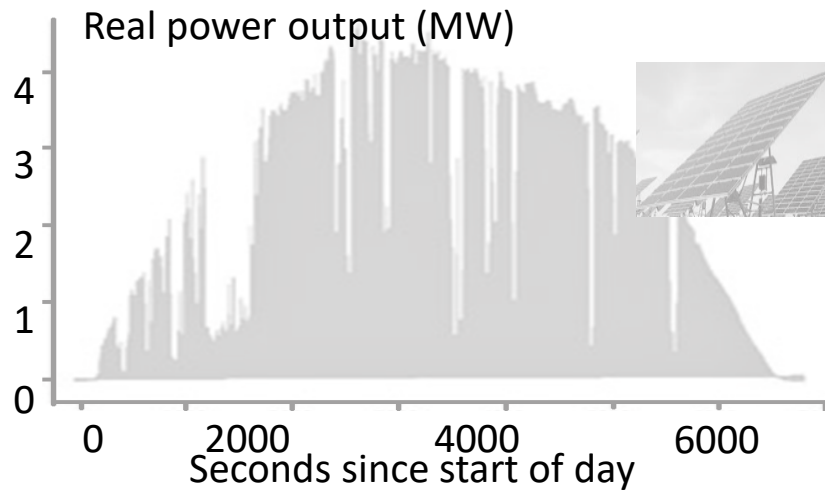
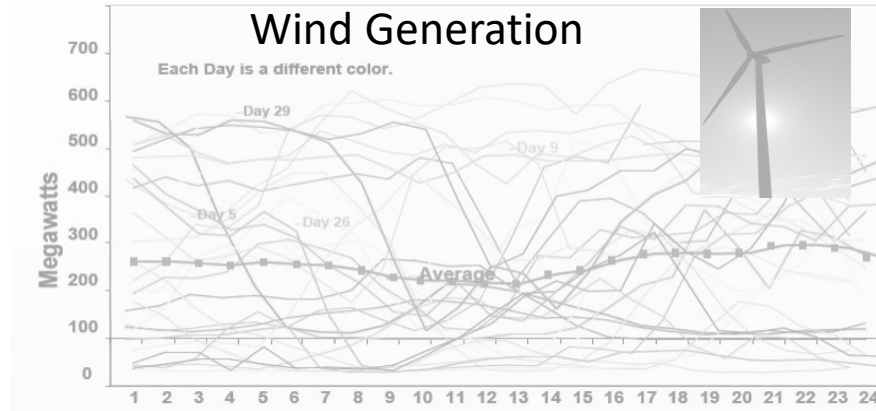


Match supply

A large blue curved arrow pointing from the 'Unresponsive' side towards the 'Responsive' side, indicating the need to match supply with demand.

Random, Intermittent

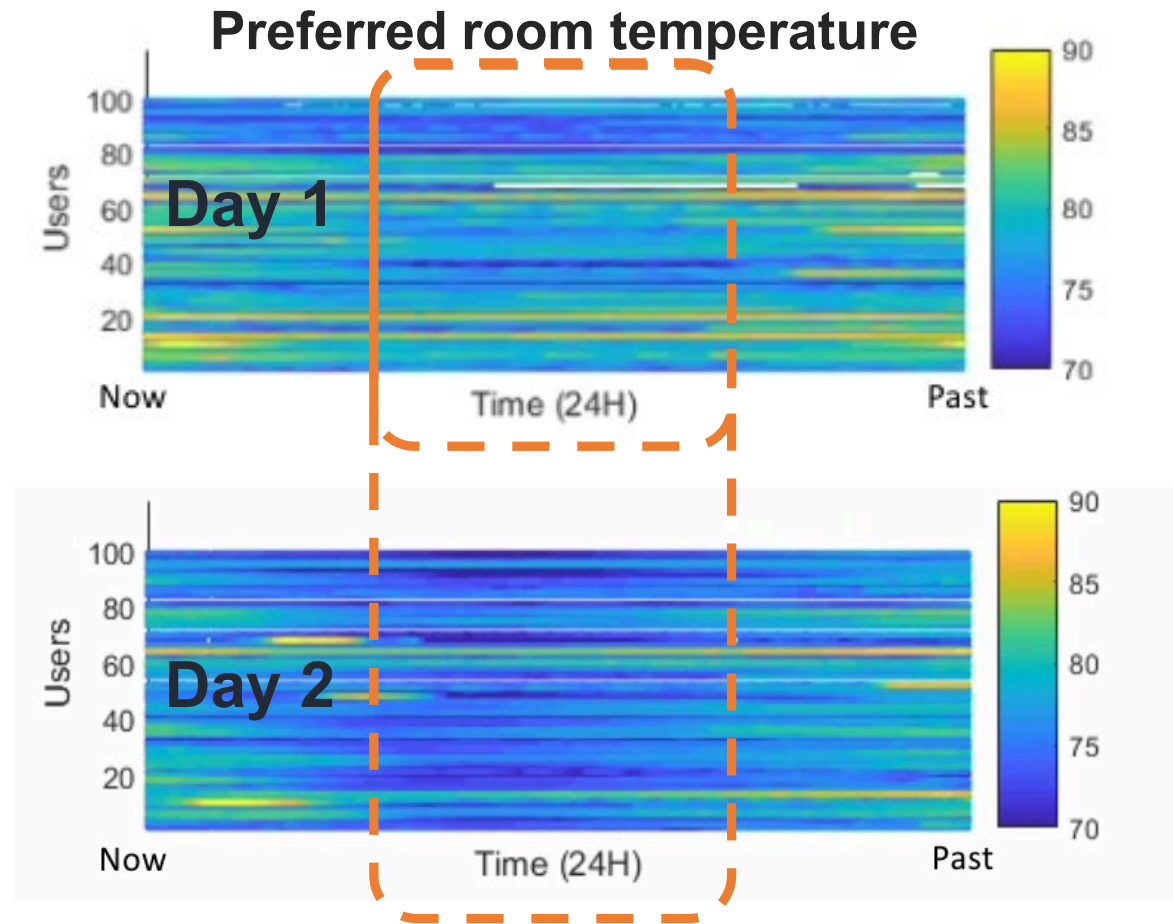
Increasing uncertainties in the grid



Random, Intermittent

??

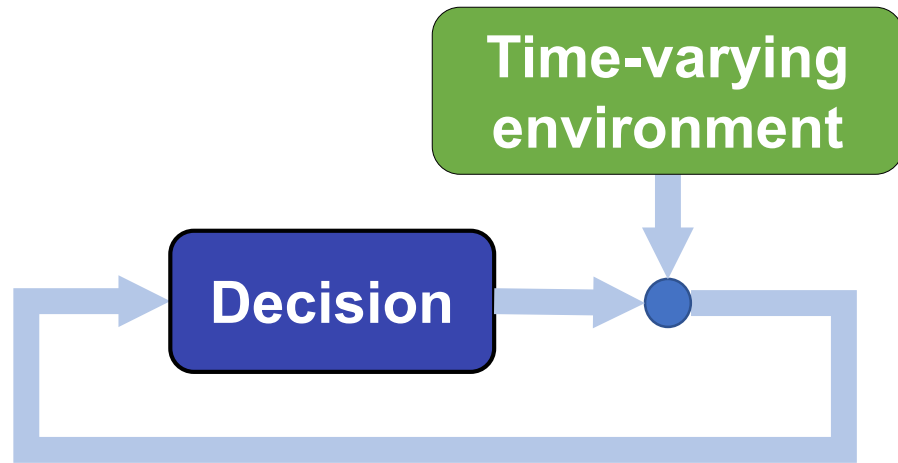
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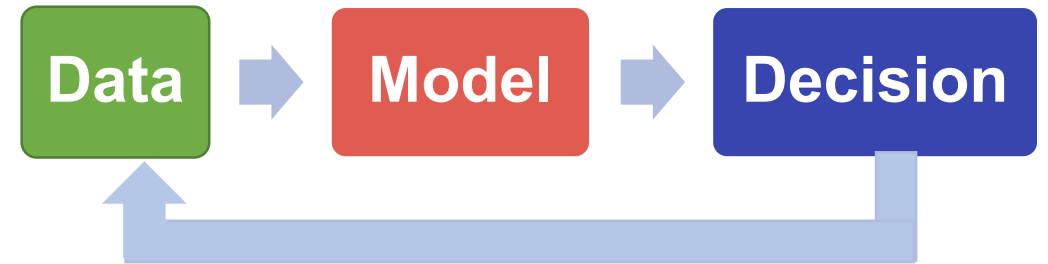
Different and **random** preferences:
Unknown to the utility/DR company

Online learning and decision making

Online decision-making under a time-varying environment

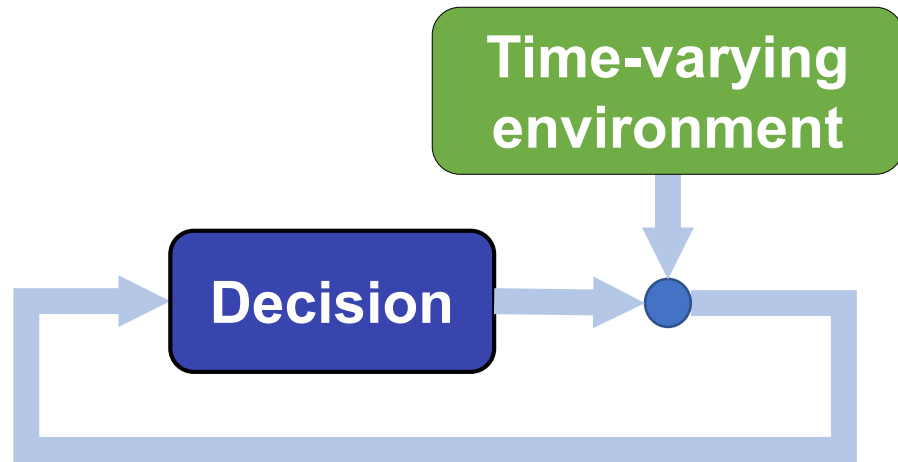


Online learning and control to learn & handle unknown uncertainties

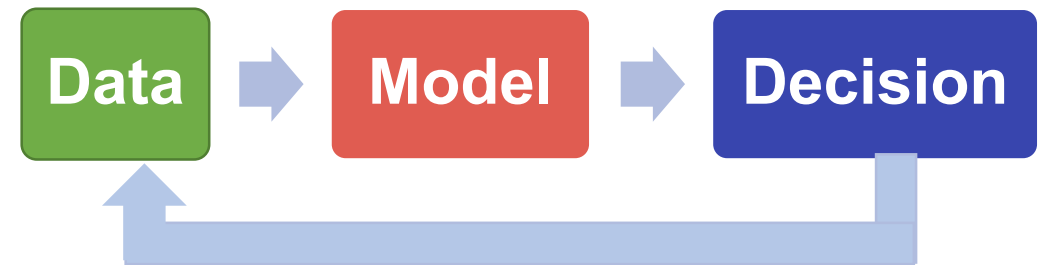


Online learning and decision making

Online decision-making under a time-varying environment

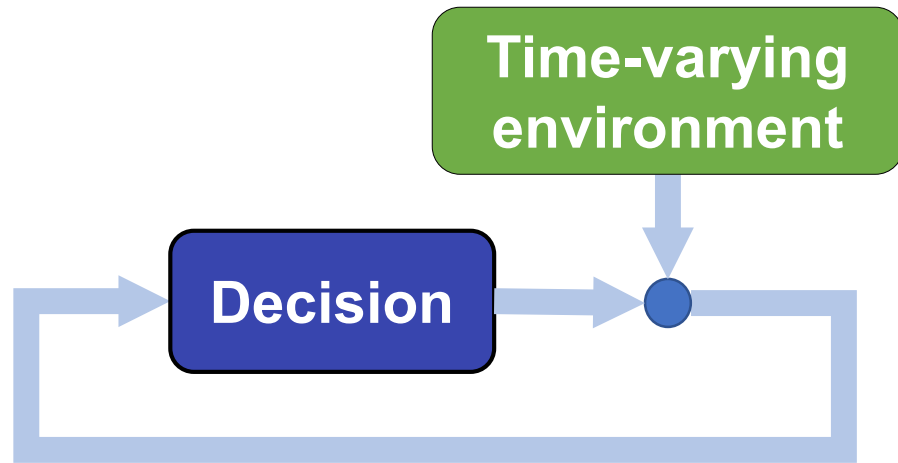


Online learning and control to learn & handle unknown uncertainties



Today's talk

Online control (online optimization with switching cost) using Prediction



Online learning and control to learn & handle unknown uncertainties

